Журнал обчислювальної та прикладної математики | **2012, № 3 (109)** с. 89–104 | Journal of Numerical & Applied Mathematics

UDC 517.958+519.6

STREAMLINE DIFFUSION SCHEMES FOR SOLVING A NONLINEAR HYPERBOLIC BOUNDARY VALUE PROBLEM

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Резюме. В роботі вивчається метод скінченних елементів для розв'язування нелінійної гіперболічної крайової задачі. З'ясовано питання існування і єдиності розв'язку, а також оцінено апріорну та апостеріорну похибки. Отримано оцінку стійкості і оптимальні порядки збіжності, показано апріорну оцінку $O(h^{k+1/2})$, де h – крок сітки і k – степінь кусково-поліноміальних функцій на скінченних елементах, в областях, де точний розв'язок є гладкий або негладкий. Для пропонованого методу наведено результати чисельних експериментів.

Abstract. In this paper we study the streamline diffusion finite element method for treating a nonlinear hyperbolic boundary value problem. The existence and uniqueness are discussed. Also, a priori and a posteriori errors are estimated for this problem. We derive the stability estimate and optimal convergence rates, showing an a priori error estimate of order $\mathcal{O}(h^{k+1/2})$ in domains where the exact solution is smooth or non-smooth; here h is the mesh width and k is the degree of the piecewise polynomial functions spanning the finite element subspaces. Also, some numerical illustrations are given for the presented method.

AMS Subject Classification: 65M12, 65M15, 65M60, 82D10, 35L80

1. Introduction

In this paper we consider the following wave equation:

$$u_{tt} - u_{xx} = \lambda F(x,t,u), \quad (x,t) \in \Omega, \tag{1}$$

$$\alpha u(t,t) - \beta \frac{\partial u}{\partial n_1}(t,t) = \alpha u(1+t, 1-t) + \beta \frac{\partial u}{\partial n_2}(1+t,1-t), \quad 0 \le t \le 1, \tag{2}$$

$$u(x,0) = 0, \qquad 0 \le x \le 2. \tag{3}$$

Where Ω is as follows:

$$\Omega = \{(x,t) : 0 \le t \le 1,\ t \le x \le 2-t\}$$

and the parameters $\lambda, \alpha, \beta \in \mathbb{R}$ such that $\alpha^2 + \beta^2 \neq 0$. The two vectors n_1 and n_2 are the exterior unit normals and $\frac{\partial u}{\partial n_1}$, $\frac{\partial u}{\partial n_2}$ are the normal derivatives. Also, $F(x,t,u) \ge 0$ and $\frac{\partial F(x,t,u)}{\partial u}$ are arbitrary continuous in Ω. The above boundary value problem for mass-spring system has an analog the continuum case which was first formulated [21, 34] as above (see also ([23, 26, 27, 28])). Our problem is a generalization of the problems studied by Kalmenov [21], and [26, 27, 28, 38]. The purpose of this paper is to present extension of the *streamline diffusion*

[†] *Key words.* Streamline diffusion method, hyperbolic problems, wave equations, error estimate, finite element.

(Sd) method to a nonlinear mass-spring system. The mathematical study of the mass spring system with this triangle domain has been considered by several authors in various settings (see [11, 22, 24, 37, 39]). One of the applications of mass spring systems to arch structure railways and long bridge-like structures reduces the dynamic and static loads due to train. Also, we can see this system to simulate facial soft tissue of great interest to many medical forms and make visible to applications([18, 20]).

Streamline diffusion ideas carry out slightly better than the different finite element methods for smooth solutions and non-smooth solutions of the first order hyperbolic problems ([32, 34, 35]) which both is higher order accurate and has good stability properties (see [2, 3, 5, 13, 14, 15, 16, 17, 19, 25, 31]). Due to the fact that the added diffusion removes oscillations near boundary layers([4, 6, 7, 8, 9, 12]). Hughes and Brooks [25] introduced this idea in the case of stationary problems. The mathematical analysis of this method for linear problems, together with extensions to time-dependent problems using space-time elements, was started in Johnson and Navert [31] and was continued in [29, 30, 33, 36]. In this paper we shall go into the details for the nonlinear hyperbolic problem and a new version of Sd method for solving the problem is given. The remaining structure of this article is organized as follows:
The uniqueness of the problem is discussed in section 2. In Section 3, we present and analyze the Sd method. In Sect. 4, by using the Sd method, we investigate stability and obtain an a priori error estimations for this system. A posteriori error estimations are given in sections 5, 6 and 7. Finally, in Sect. 8 the paper would be completed by the inclusion of numerical results to provide experimental support for the theoretical results and show how the method performs in practice.

2. Existence and Uniqueness of the solution

In the following propositions, it is shown that there is a unique solution for (1)-(3) in linear and nonlinear form for $F(x,t,v)$ in Sobolev space ([1]) of $W_2^1(\Omega) \cap W_2^1(\partial\Omega) \cap C(\overline{\Omega})$. In [28] we observe that the linear problem is considered and in [42] existence theorems for some nonlinear hyperbolic equations are given, but in this section the uniqueness of nonlinear form is studied.

Proposition 1. *For $k = 0, 1, 2, ...$ given $\lambda, \alpha, \beta \in \mathbb{R}$ and $F \in H^k(\Omega)$, problem of* (1)-(3) *has a unique solution in the Hilbert space $u \in H^{k+2}(\Omega)$.*

Proof. We extend the proof of theorem's Iraniparst (see [28]) and we use some propositions and lemmas in [42] (see 2.3 and 4.3). We influence the change of variables $X = x - t$ and $Y = x + t$ into (1)-(3). Hence, we have

$$V_{XY} = \gamma\widehat{F}(X, Y, V(X, Y)), \quad (X, Y) \in \Omega', \tag{4}$$

$$\Omega' = \{(X, Y) : 0 \leq Y \leq 2, \quad 0 \leq X \leq Y\}$$

$$\alpha V(0, Y) + \beta V_X(0, Y) = \alpha V(Y, 2) + \beta V_Y(Y, 2), \quad 0 \leq Y \leq 2$$

$$V(X, X) = 0, \quad 0 \leq X \leq 2,$$

where $\widehat{F}(X,Y,V(X,Y)) = F(\frac{x+y}{2}, \frac{-x+y}{2}, u(\frac{x+y}{2}, \frac{-x+y}{2}))$ and $\gamma = \frac{-\lambda}{4}$. Integrating Eq. (4) and using the above boundary conditions we have

$$\begin{aligned} V(\xi,\eta) =& \gamma(\int\int_{\Omega'} G(\xi,\eta;X,Y)\widehat{F}(X,Y,V(X,Y))\,dX\,dY \\ &- \int_0^2 r(\xi,\eta,X)(\widehat{F}(0,X,V(0,X)) - \widehat{F}(2,X,V(X,2))dX), \end{aligned}$$

such that η, $\xi \in \Omega'$ and

$$r(\xi,\eta,X) = \begin{cases} 0 & \text{if } 0 \le X \le \xi \\ \beta/(2\alpha) & \text{if } \xi \le X \le \eta \\ 0 & \text{if } \eta \le X \le 2. \end{cases}$$

In [21, 34, 35, 28] the Green's function, $G(\xi,\eta;X,Y)$ described. Also, in [42, 35, 28] we observe that the critical eigenvalues are extended based on spectral theory (see section 2 [42]). □

Proposition 2. *If $F(x,t,v) = \int_\Omega k(x-t)v(x,t)d\Omega = k * v$ and we have the above assumptions in proposition of 1 then problem (1)-(3) has a unique solution.*

Proof. By using the above proposition, [26, 27, 28, 42] and the Hilbert translations the proof is completed. □

3. The Streamline Diffusion Method

For simplify in (1)-(3) we assume $\lambda = 1$. We introduce variables $v = \partial u/\partial t$ and $\dot{v} = \partial v/\partial t$. Hence, we rewrite (1)-(3) to

$$\begin{cases} L\mathbf{w} \equiv \dot{\mathbf{w}}(x,t) + A\mathbf{w}(x,t) = f(u) & \text{in } \Omega \\ \mathbf{w}(x,0) = \mathbf{0}, & 0 \le x \le 2 \\ B\mathbf{w'}(t,t) = C\mathbf{w''}(1+t, 1-t), & 0 \le t \le 1. \end{cases} \tag{5}$$

Here, we assume that $\mathbf{w}(x,t) = (u(x,t), v(x,t))^T$, $\dot{\mathbf{w}}(x,t) = (\dot{u}(x,t), \dot{v}(x,t))^T$, $\mathbf{w'} = (u, \frac{\partial u}{\partial n_1})^T$, $\mathbf{w''} = (u, \frac{\partial u}{\partial n_2})^T$, $A = \begin{pmatrix} 0 & -1 \\ -\frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}$, $B = \begin{pmatrix} \alpha & -\beta \end{pmatrix}$, $C = \begin{pmatrix} \alpha & \beta \end{pmatrix}$ and $f(u) = (0, F(x,t,u))^T$.

In this section, we consider the Sd-method for solving (5). In this method, instead of using the standard Galerkin method, is usual in Finite Element Method, for the one variable (spatial or time) we used the Galerkin method simultaneously in space and time. That is, we use finite element and interpolation functions depend on time and space.

Space-time Sd-method can be used to improve stabilization, however used without care, this would lead to a very large linear system to be solved. One of the reasons for it is that in this technique the use of continuous (in time) test and trial functions in all levels of time. One way to avoid this difficultly, and decrease the size of the corresponding linear system, is to work in slabs of space-time, with the help of interpolation functions that will be continuous in the spatial variables but will be discontinuous in the time variables at the common frontier of every two slabs.

Sd-method for (5) is based on using finite element over the space-time domain Ω. To define this method, let $0 = t_0 < t_1 < \cdots < t_N = 1$ be a subdivision of the time interval $[0,1]$ into intervals $I_n = (t_n, t_{n+1})$, with time steps $k_n = t_{n+1} - t_n$, $n = 0, 1, \cdots, N-1$ and introduce the corresponding space-time slabs (see Fig. 1.), i. e.,

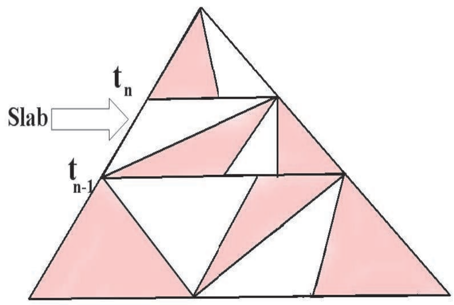

FIG. 1. The slabs on Ω

$$S_n = \Big\{(x,t) : \quad t_{n+1} \le x \le 2 - t_{n+1}, \qquad t \le x \le t_{n+1},$$
$$2 - t_{n+1} \le x \le 2 - t, \qquad t_n < t < t_{n+1}\Big\},$$

for $n = 0, 1, ..., N-2$ and

$$S_{N-1} = \{(x,t) : t \le x \le 2 - t, \quad t_{N-1} < t < t_N\}.$$

Further, for each n let $\mathbf{W}^{n,\alpha,\beta}$ be a finite element subspace of $H^1(S_n) \times H^1(S_n)$, based on triangulation of the slab S_n with elements of size h and let

$$\dot{\mathbf{W}}^{n,\alpha,\beta} = \Big\{\mathbf{w} \in \mathbf{W}^{n,\alpha,\beta} | B\mathbf{w}'(t,t) = C\mathbf{w}''(t+1, 1-t), \quad 0 \le t \le 1\Big\}.$$

Simplifying, we get boundary condition in $\dot{\mathbf{W}}^{n,\alpha,\beta}$ equal zero. We can formulate Sd-method on the slab S_n for (5), as follows:
For $n = 0, \cdots, N-1$, find $\mathbf{w}^n \in \dot{\mathbf{W}}^n$ such that

$$(\dot{\mathbf{w}}^{n,\alpha,\beta} + A\mathbf{w}^{n,\alpha,\beta}, g + \delta(\dot{g} + Ag))_n + \langle \mathbf{w}_+^n, g_+\rangle_n + \langle \mathbf{w}_+^{n,\alpha,\beta}, g_+\rangle_{\Gamma_n} = \tag{6}$$

$$= (f(u^n), g + \delta(\dot{g} + Ag))_n + \langle \mathbf{w}_-^{n,\alpha,\beta}, g_+\rangle_n.$$

We have $g + \delta(\dot{g} + Ag)$, as a test function such that $\delta = \bar{C}h$ with $\bar{C}$ is a suitable chosen (sufficiently small, see [33]) positive constant. Further, we define the following notations for (6) and everywhere in the paper:

$$(\mathbf{u},\mathbf{v})_n = \int_{S_n} \mathbf{u}^T.\mathbf{v} dxdt,$$

$$(\mathbf{u},\mathbf{u})_n = \|\mathbf{u}\|_n^2,$$

$$\langle \mathbf{u},\mathbf{v}\rangle_n = \int_{t_n}^{2-t_n} \mathbf{u}^T(x,t_n).\mathbf{v}(x,t_n)dx,$$
$$\langle \mathbf{u},\mathbf{u}\rangle_n = |\mathbf{u}|_n^2,$$
$$\mathbf{v}_+(x,t) = \lim_{s\to 0^+} \mathbf{v}(x,t+s),$$
$$\mathbf{v}_-(x,t) = \lim_{s\to 0^-} \mathbf{v}(x,t+s),$$
$$\langle \mathbf{u}_+,\mathbf{v}_+\rangle_\Gamma = \int_\Gamma \mathbf{u}_+{}^T.\mathbf{v}_+ d\sigma,$$
$$\langle \mathbf{u}_+,\mathbf{v}_+\rangle_{\Gamma_n} = \int_{\Gamma_n} \mathbf{u}_+{}^T.\mathbf{v}_+ ds,$$

also $\|.\| = \|.\|_{L_2(\Omega)}$, $\|.\|_{\infty,\Omega} = \|.\|_{L_\infty(\Omega)}$, $\|.\|_s = \|.\|_{s,\Omega} = \|.\|_{H^s(\Omega)}$, $\Gamma = \partial\Omega$ and $\Gamma = \bigcup_{n=0}^{N-1}\Gamma_n$. The terms including $\langle \quad , \quad \rangle_{\Gamma,\Gamma_n}$ in the above formula is a jump conditions which imposes a weakly enforced continuity condition across the slab interfaces, at t_n and is the mechanism by which information is propagated from one slab to another. For more concisely, after summing over n and $f(\varpi) \simeq f(g) + (\varpi - g).\frac{\partial f(g)}{\partial u}$ (such that $\varpi = (\ u0\ )$), we get the function space $\prod_{n=0}^{N-1} \dot{\mathbf{W}}^{n,\alpha,\beta}$, therefore we may rewrite (6) as follow:

find $\mathbf{w} \in \prod_{n=0}^{N-1} \dot{\mathbf{W}}^{n,\alpha,\beta}$, such that

$$B(\mathbf{w},g) = L(g), \tag{7}$$

for $g \in \prod_{n=0}^{N-1} \dot{\mathbf{W}}^{n,\alpha,\beta}$. The bilinear form $B(.,.)$ and the linear form $L(.)$ are defined by

$$B(\mathbf{w},g) = \sum_{n=0}^{N-1}\{(\dot{\mathbf{w}}^{n,\alpha,\beta} + A\mathbf{w}^{n,\alpha,\beta} - \varpi^n.\frac{\partial f}{\partial u}(g), g + \delta(\dot{g} + Ag))_n + \langle \mathbf{w}_+^{n,\alpha,\beta}, g_+\rangle_{\Gamma_n}\}$$
$$+ \sum_{n=1}^{N-1}\langle[\mathbf{w}^{n,\alpha,\beta}], g_+\rangle_n + \langle \mathbf{w}_+^{n,\alpha,\beta}, g_+\rangle_0,$$
$$L(g) = \sum_{n=0}^{N-1}(f(g) - g.\frac{\partial f}{\partial u}(g), g + \delta(\dot{g} + Ag))_n$$

for $\mathbf{w} = (\mathbf{w}_1,\mathbf{w}_2)^T$ and $\varpi = (\mathbf{w}_1, 0)^T$. Also, we assume that $[\mathbf{w}_i] = \mathbf{w}_{i,+} - \mathbf{w}_{i,-}$, for $i = 1,2$, $[\mathbf{w}] = ([\mathbf{w}_1],[\mathbf{w}_2])^T$. Let T_h^n be a triangulation of the slab S_n into triangles K, for $h > 0$, and introduce

$$\mathbf{W}_h^{n,\alpha,\beta} = \Big\{\mathbf{w} \in \dot{\mathbf{W}}^{n,\alpha,\beta} : \mathbf{w}|_K \in [P_k(K)] \times [P_k(K)] \subseteq H^1(S_n) \times H^1(S_n), \quad K \in T_h^{n,\alpha,\beta}\Big\}$$

where $P_k(K)$ denotes the set of polynomials in K of degree less than or equal k and

$$\mathbf{W}_h = \prod_{n=0}^{N-1} \mathbf{W}_h^{n,\alpha,\beta}.$$

Thus (7) can be formulated as follows:

Find $\mathbf{w}_h = \begin{pmatrix} u_h \\ v_h \end{pmatrix} \in \mathbf{W}_h$ such that

$$B(\mathbf{w}_h, g) = L(g), \tag{8}$$

for $g \in \mathbf{W}_h$. Moreover, we know that the exact solution of (7) satisfies

$$B(\mathbf{w}, g) = L(g),$$

for $g \in \dot{\mathbf{W}}^{n,\alpha,\beta}$, and by subtraction we have the following error equation

$$B(e, g) = 0, \tag{9}$$

where $e = \mathbf{w} - \mathbf{w}_h$ and $\mathbf{w} \in \mathbf{W}_h$.

4. Stability for the Sd-Method

Below, we derive the stability estimate for Sd-method (7). These estimate will be of crucial importance in proving the finite element analysis. We apply properties of the bilinear $B(.,.)$ and obtain stability estimate. For our problem, we have the following stability Proposition:

Proposition 3. *For any* $\boldsymbol{w} = \begin{pmatrix} u \\ v \end{pmatrix} \in \prod_{n=0}^{N-1} \boldsymbol{W}^{n,\alpha,\beta}$ *with assumptions* $uv \leq 0$ *and* $\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} \geq 0$ *we have:*

$$B(\boldsymbol{w}, \boldsymbol{w}) \geq |||\boldsymbol{w}|||^2 = \frac{1}{2}\{\mid \boldsymbol{w}_- \mid_N^2 - \mid \boldsymbol{w}_+ \mid_0^2 + \delta \parallel \dot{\boldsymbol{w}} + A\boldsymbol{w} \parallel_\Omega^2\} + \mid \boldsymbol{w}_+ \mid_\Gamma^2 . \tag{10}$$

Proof. Using the definition of the bilinear form B and setting $g = \mathbf{w}$ it follows:

$$B(\mathbf{w}, \mathbf{w}) = (\dot{\mathbf{w}}, \mathbf{w})_\Omega + (A\mathbf{w}, \mathbf{w})_\Omega + \delta \parallel \dot{\mathbf{w}} + A\mathbf{w} \parallel_\Omega^2 + \mid \mathbf{w}_+ \mid_\Gamma^2 + \\ + \sum_{n=1}^{N-1} \langle [\mathbf{w}], \mathbf{w}_+ \rangle_n + \langle \mathbf{w}_+, \mathbf{w}_+ \rangle_0.$$

Integrating by parts yields

$$(\dot{\mathbf{w}}, \mathbf{w})_\Omega + \sum_{n=1}^{N-1} \langle [\mathbf{w}], \mathbf{w}_+ \rangle_n + \langle \mathbf{w}_+, \mathbf{w}_+ \rangle_0 = \\ = \frac{1}{2}\{\mid \mathbf{w}_- \mid_N^2 + \mid \mathbf{w}_+ \mid_0^2 + \sum_{n=1}^{N-1} \mid [\mathbf{w}] \mid_n^2\}.$$

Therefore, by using the assumptions of the proposition the proof is complete. □

We use the standard argument for finite element and introduce the linear nodal interpolate $I_h\mathbf{w} \in W_h$ of the exact solution $\mathbf{w}$ and we set $\zeta = \mathbf{w} - I_h\mathbf{w}$, $\xi = \mathbf{w}_h - I_h\mathbf{w}$. Thus, we have:

$$e := \mathbf{w} - \mathbf{w}_h = (\mathbf{w} - I_h\mathbf{w}) - (-I_h\mathbf{w} + \mathbf{w}_h) = \zeta - \xi.$$

Recalling the Galerkin orthogonality relation (9):

$$B(e, \mathbf{w}) = 0. \tag{11}$$

Now, we can prove the basic global error estimate by using proposition 3.

Proposition 4. *If* $\boldsymbol{w}_h \in \boldsymbol{W}_h$ *satisfies in* (8) *and* $\boldsymbol{w}$ *is exact solution converted mass-spring* (5)*, and also*

$$\| A \|_{\infty,\Omega} \leq C,$$

then, there is a constant C *such that*

$$|||\boldsymbol{w} - \boldsymbol{w}_h||| \leq Ch^{k+1/2} \|\boldsymbol{w}\|_{k+1}.$$

Proof. Using the basic stability estimate (10) with $\mathbf{w} = e$ and (11), with $\mathbf{w} = \xi$, we get that

$$|||e|||^2 \leq B(e,e) = B(e,\zeta) - B(e,\xi) = B(e,\zeta) =$$
$$= (\dot{e} + Ae, \zeta + \delta(\dot{\zeta} + A\zeta))_\Omega + \sum_{n=0}^{N-1} \langle [e], \zeta_+ \rangle_n + \langle e_+, \zeta_+ \rangle_\Gamma.$$

Moreover, we use the inequality $2ab \leq \epsilon a^2 + \epsilon^{-1} b^2$ for a, b real numbers and $\epsilon > 0$. Therefore, we have:

$$B(e,\zeta) \leq \frac{\delta}{8} \| \dot{e} + Ae \|_\Omega^2 + \frac{2}{\delta} \| \zeta \|_\Omega^2 + \frac{\delta}{8} \| \dot{e} + Ae \|_\Omega^2 + 2\delta \| \dot{\zeta} + A\zeta \|_\Omega^2$$
$$+ \frac{1}{4} \sum_{n=1}^{N-1} | [e] |_n^2 + \sum_{n=1}^{N-1} | \zeta_+ |_n^2 + \frac{1}{4} | e_+ |_0^2 + | \zeta_+ |_0^2 + \frac{1}{4} \| e_+ \|_\Gamma^2 + \| \zeta_+ \|_\Gamma^2 .$$

According to the above proposition and (10), we can write

$$B(e,\zeta) \leq \frac{1}{4} |||e|||^2 +$$
$$+ \left\{ \frac{2}{\delta} \| \zeta \|_\Omega^2 + 2\delta \| \dot{\zeta} + A\zeta \|_\Omega^2 + \sum_{n=1}^{N-1} | \zeta_+ |_n^2 + | \zeta_+ |_0^2 + \| \zeta_+ \|_\Gamma^2 \right\}.$$

On the other hand, we have the inequality

$$\| \dot{\zeta} + A\zeta \|_\Omega \leq \| \dot{\zeta} \|_\Omega + \| A \|_{\infty,\Omega} \| \zeta \|_\Omega . \tag{12}$$

With using inverse estimate inequality, we have

$$\| \dot{\zeta} \|_\Omega \leq Ch^{-1} \| \zeta \|_\Omega . \tag{13}$$

Therefore, with (12), (13) and assumption $\delta = \bar{C}h$, we obtain:

$$|||e|||^2 \leq C \left\{ \| \zeta_+ \|_\Gamma^2 + h^{-1} \| \zeta \|_\Omega^2 + \sum_{n=0}^{N-1} | \zeta_+ |_n^2 + h \| \zeta \|_{1,\Omega}^2 \right\}.$$

Finally, by standard interpolation theory it follows that (see e.g. Ciarlet [12])

$$\left[h \| \zeta_+ \|_\Gamma^2 + \| \zeta \|_\Omega^2 + h \sum_{n=0}^{N-1} | \zeta_+ |_n^2 + h^2 \| \zeta \|_{1,\Omega}^2 \right]^{1/2} \leq Ch^{k+1} \| \mathbf{w} \|_{k+1,\Omega},$$

which proves the desired estimates. □

We observe in the remarked references that the corresponding optimal convergence rate for the popular numerical methods in the literatures such as conservative finite difference method, semi-implicit finite difference method, semi-discrete finite element method, the time-splitting spectral method or Galerkin method are of order $\mathcal{O}(h^k)$.

5. An a posteriori error estimate

In this section, we shall consider the following simplified version of Sd-method for (6) and (8) with $\delta = 0$:
Find $\mathbf{w}_h \in \mathbf{W}_h$, such that for $n = 0, 1, ..., N-1$:

$$(\dot{\mathbf{w}}_h + A\mathbf{w}_h, g)_n + \langle [\mathbf{w}_h], g_+ \rangle_n = (f, g)_n, \qquad \forall g \in \mathbf{W}_h, \tag{14}$$

where $[\mathbf{w}_h] = \mathbf{w}^n_{h,+} - \mathbf{w}^n_{h,-}$ and $\mathbf{w}^0_{h,-} = 0$.
In order to obtain a representation of the error, we consider the following auxiliary problem, referred to as the linearized dual problem:
Find Φ such that

$$\begin{cases} L^*\Phi \equiv -\Phi_t + A^T\Phi = \psi^{-1}e, & \text{in} \quad \Omega, \\ \Phi(t,t) = 0, & t \in [0,1], \\ \Phi(1+t, 1-t) = 0, & t \in [0,1], \\ \Phi(x,1) = 0, & x \in [0,2] \end{cases} \tag{15}$$

and L^* denotes the adjoint of the operator L defined in (15) and ψ is a positive weight function. Note that this problem is computed "backward", but there is a corresponding change in sign. Further, we shall introduce the following notation:

$$\| e \|_{L_2^{\psi}(\Omega)} = (e, \psi e)_{\Omega}^{1/2}. \tag{16}$$

Multiplying (15) by e and integrating by parts, and summing over n, we obtain the following error representation formula:

$$\| e \|^2_{L_2^{\psi^{-1}}(\Omega)} = (e, \psi^{-1}e)_{\Omega} = (e, L^*\Phi) \tag{17}$$

$$= \sum_{n=0}^{N-1} (e, -\Phi_t + A^T\Phi)_n = \sum_{n=0}^{N-1} (e, -\Phi_t)_n + \sum_{n=0}^{N-1} (e, A^T\Phi)_n.$$

On the other hand, we have for $n = 0, 1, ..., N-2$:

$$(e, -\Phi_t)_n = \int_{S_n} (-e^T.\Phi_t)dxdt = \int_{t_n}^{t_{n+1}} \int_{t_n}^{x} (-e^T.\Phi_t)dtdx \tag{18}$$

$$+ \int_{t_{n+1}}^{2-t_{n+1}} \int_{t_n}^{t_{n+1}} (-e^T.\Phi_t)dtdx$$

$$+ \int_{2-t_{n+1}}^{2-t_n} \int_{t_n}^{2-x} (-e^T.\Phi_t)dtdx$$

$$= (e_t, \Phi)_n + \int_{t_n}^{2-t_n} e^T(x, t_n).\Phi(x, t_n)dx - \int_{t_{n+1}}^{2-t_{n+1}} e^T(x, t_{n+1}).\Phi(x, t_{n+1})dx,$$

and for $n = N-1$:

$$(e, -\Phi_t)_{N-1} = (e_t, \Phi)_{N-1} + \int_{t_{N-1}}^{2-t_{N-1}} e^T(x, t_{N-1}).\Phi(x, t_{N-1})dx. \quad (19)$$

Hence, if we assume $e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ and $\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$ then, we obtain for $n = 0, 1, ..., N-1$:

$$(e, A^T\Phi)_n = \int_{S_n} e^T.A^T\Phi dxdt$$

$$= \int_{S_n} e^T.\begin{pmatrix} 0 & -\frac{\partial^2}{\partial x^2} \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} dxdt = \int_{S_n} (e_1, e_2).\begin{pmatrix} \frac{-\partial^2\Phi_2}{\partial x^2} \\ -\Phi_1 \end{pmatrix} dxdt.$$

Therefore by parts integrating and same as (18), we have:

$$\int_{S_n} (-e_1\frac{\partial^2\Phi_2}{\partial x^2} - e_2\Phi_1)\, dx\, dt = \int_{S_n} (-\Phi_2\frac{\partial^2 e_1}{\partial x^2} - e_2\Phi_1)dxdt = (Ae, \Phi)_n. \quad (20)$$

By using (18) and (19) in the following definition, we have:

$$J = \sum_{n=0}^{N-2}\left(\int_{t_n}^{2-t_n} e^T(x, t_n).\Phi(x, t_n)dx - \int_{t_{n+1}}^{2-t_{n+1}} e^T(x, t_{n+1}).\Phi(x, t_{n+1})dx\right) +$$

$$+ \int_{t_{N-1}}^{2-t_{N-1}} e^T(x, t_{N-1}).\Phi(x, t_{N-1})dx =$$

$$= (\langle e_-, \Phi_-\rangle_1 - \langle e_+, \Phi_+\rangle_0) + (\langle e_-, \Phi_-\rangle_2 - \langle e_+, \Phi_+\rangle_1) + ...$$

$$+(\langle e_-, \Phi_-\rangle_{N-1} - \langle e_+, \Phi_+\rangle_{N-2}) + (\langle e_-, \Phi_-\rangle_N - \langle e_+, \Phi_+\rangle_{N-1}).$$

We rearrange the above summation by putting $\Phi_- = \Phi_- - \Phi_+ + \Phi_+$, then we can write:

$$J = \langle e_-, \Phi_-\rangle_N + \langle e_+, \Phi_+\rangle_0 + \sum_{n=0}^{N-1}\langle [e], \Phi_+\rangle_n + \sum_{n=0}^{N-1}\langle e_-, [\Phi]\rangle_n.$$

According to (15), $\Phi(., t_N = 1) = 0$ and since $e_-^0 = [\mathbf{w}^0] = 0$, we get

$$J = \sum_{n=0}^{N-1}\langle [\mathbf{w}_h], \Phi_+\rangle_n. \quad (21)$$

Therefore by replacing (18)-(21) in (17), we have:

$$\| e \|^2_{L_2^{\psi^{-1}}(\Omega)} = \sum_{n=0}^{N-1}(e_t, \Phi) + \sum_{n=0}^{N-1}(Ae, \Phi)_n - \sum_{n=0}^{N-1}\langle [\mathbf{w}_h], \Phi_+\rangle_n$$

$$= \sum_{n=0}^{N-1}((\mathbf{w} - \mathbf{w}_h)_t + A(\mathbf{w} - \mathbf{w}_h), \Phi)_n - \sum_{n=0}^{N-1}\langle [\mathbf{w}_h], \Phi_+\rangle_n$$

$$= \sum_{n=0}^{N-1}(f - \mathbf{w}_{h,t} - A\mathbf{w}_h, \Phi)_n - \sum_{n=0}^{N-1}\langle [\mathbf{w}_h], \Phi_+\rangle_n.$$

Hence, with recalling (5) and using the Galerkin orthogonality, we obtain

$$\begin{aligned}\| e \|^2_{L_2^{\psi^{-1}}(\Omega)} = & \sum_{n=0}^{N-1} (f - \mathbf{w}_{h,t} - A\mathbf{w}_h, \hat{\Phi} - \Phi)_n - \\ & - \sum_{n=0}^{N-1} \langle [\mathbf{w}_h], (\hat{\Phi} - \Phi)_+ \rangle_n \equiv I + II.\end{aligned} \tag{22}$$

Where $\hat{\Phi} \in W_h$ is an interpolation of Φ. The idea is now to estimate $\hat{\Phi} - \Phi$ in terms of $\psi^{-1}e$ using a strong stability estimates for solution Φ of the dual problem.

6. Interpolation estimates

In the following we consider two L_2-projections for $\hat{\Phi} \in \mathbf{W}_h$ in (9):

$$P_n : L_2([0,2]) \longmapsto \mathbf{W}_h^n,$$

$$\pi_n : L_2(S_n) \longmapsto \Pi_{0,n} = \{\mathbf{w} \in L_2(S_n) : \mathbf{w}(x,.) \text{ is constant on } I_n, x \in [0,2]\},$$

such that

$$\int_0^2 (P_n\Phi)^T.\mathbf{w}dx = \int_0^2 \Phi^T.\mathbf{w}dx, \qquad \forall \mathbf{w} \in \mathbf{W}_h^n,$$

$$\pi_n \mathbf{w} \mid_{S_n} = \frac{1}{k_n}\int_{I_n} \mathbf{w}(.,t)dt, \quad \forall \mathbf{w} \in \Pi_{0,n}.$$

Then, we can define $\hat{\Phi} \mid_{S_n} \in \mathbf{W}_h^n$ by letting

$$\hat{\Phi} \mid_{S_n} = P_n\pi_n\Phi = \pi_n P_n \Phi \in \mathbf{W}_h^n,$$

where $\Phi = \Phi \mid_{S_n}$ and we can observe that P_n and π_n are commuted. Moreover, if we introduce P and π defined by

$$(P\Phi) \mid_{S_n} = P_n(\Phi \mid_{S_n}),$$

and

$$(\pi\Phi) \mid_{S_n} = \pi_n(\Phi \mid_{S_n}),$$

then we can put :

$$\hat{\Phi} = P\pi\Phi = \pi P\Phi \in \mathbf{W}_h.$$

Now, we define the following residuals:

$$R_0 = f - \mathbf{w}_{h,t} - A\mathbf{w}_h,$$

$$R_1 = \frac{\mathbf{w}^n_{h,+} - \mathbf{w}^n_{h,-}}{k_n}, \qquad \text{on} \quad S_n,$$

$$R_2 = \frac{(P_n - I)\mathbf{w}^n_{h,-}}{k_n}, \qquad \text{on} \quad S_n,$$

where I is the identity operator.

In the end of this section, we shall give a lemma for some interpolation estimates by the projection operators P, leaving the overall of I and II to next section.

Lemma 1. *There is a constant C such that for residual $R \in L_2(\Omega)$,*

$$|(R, \Phi - P\Phi)_\Omega| \leq C \parallel h^2(I-P)R \parallel_{L_2^{\psi^{-1}}(\Omega)} \parallel \Phi_{xx} \parallel_{L_2^{\psi}(\Omega)} . \tag{23}$$

Proof. See [41] and [40]. □

7. The completion of the proof of a posteriori error estimates

In this section we state and prove a posteriori error estimate by estimating of the terms I and II in the error representation formula (22). To this approach we introduce the stability factors(see [10]) associated with discretization in time and space, defined by

$$Y_e^t = \frac{\parallel \Phi_t \parallel_{L_2^{\psi}(\Omega)}}{\parallel e \parallel_{L_2^{\psi^{-1}}(\Omega)}}, \tag{24}$$

and

$$Y_e^x = \frac{\parallel \Phi_{xx} \parallel_{L_2^{\psi}(\Omega)}}{\parallel e \parallel_{L_2^{\psi^{-1}}(\Omega)}} \tag{25}$$

respectively. We now apply the result of the previous sections; using Cauchy-Schwartz inequality in (22) coupled with the interpolation estimate (23) and the strong stability factors (24) and (25), to derive the $L_2(L_2)$ a posteriori error estimates for the scheme (14).

Proposition 5. *The error $e = \mathbf{w} - \mathbf{w}_h$, where $\mathbf{w}$ is the solution of the continuous problems* (5) *and $\mathbf{w}_h$ that of* (14)*, satisfies the following stability estimate:*

$$\parallel e \parallel_{L_2^{\psi^{-1}}(\Omega)} \leq CY_e^x \parallel h^2(I-P)R_0 \parallel_{L_2^{\psi^{-1}}(\Omega)} + CY_e^t \parallel k_n R_1 \parallel_{L_2^{\psi^{-1}}(\Omega)} +$$
$$+ Y_e^x \parallel h^2 R_2 \parallel_{L_2^{\psi^{-1}}(\Omega)} + Y_e^t \parallel k_n R_2 \parallel_{L_2^{\psi^{-1}}(\Omega)} .$$

Proof. Using the notation introduce above, we may write (22) as

$$\parallel e \parallel^2_{L_2^{\psi^{-1}}(\Omega)} = \sum_{n=0}^{N-1} (R_0, \hat{\Phi} - \Phi)_n + \sum_{n=0}^{N-1} \langle k_n \frac{[\mathbf{w}_h]}{k_n}, (\hat{\Phi} - \Phi)_+ \rangle_n = I + II.$$

Below we shall estimate the terms I and II separately. Splitting the interpolation error by writing $\hat{\Phi} - \Phi = \hat{\Phi} - P\Phi + P\Phi - \Phi$ and $\hat{\Phi}_n = \pi_n P\Phi$, we have:

$$I = \sum_{n=0}^{N-1} (R_0, \hat{\Phi}_n - P\Phi + P\Phi - \Phi)_n = \sum_{n=0}^{N-1} (R_0, \hat{\Phi}_n - P\Phi)_n +$$
$$+ \sum_{n=0}^{N-1} (R_0, P\Phi - \Phi)_n \leq C \parallel h^2(I-P)R_0 \parallel_{L_2^{\psi^{-1}}(\Omega)} \parallel \Phi_{xx} \parallel_{L_2^{\psi}(\Omega)} .$$

It remains to estimate the term II, to this end, we consider the following notation:

$$\Phi_+^n(x) = \Phi(x,t) - \int_{t_n}^{t} \frac{\partial}{\partial \tau} \Phi(x, \tau) d\tau,$$

hence, with integrating over I_n, we have:

$$k_n\Phi_+^n(x) = \int_{I_n} \Phi(x,t)dt - \int_{I_n}\int_{t_n}^{t} \Phi_\tau(x,\tau)d\tau dt \tag{26}$$

where $\Phi_\tau = \frac{\partial\Phi}{\partial\tau}$ and $\Phi^n = \Phi(.,t_n)$.

$$II = \sum_{n=0}^{N-1}\langle k_n\frac{[\mathbf{w}_h]}{k_n}, (\hat{\Phi}-\Phi)_+\rangle_n = \sum_{n=0}^{N-1}\langle k_n\frac{[\mathbf{w}_h]}{k_n}, (\hat{\Phi}_n - P\Phi + P\Phi - \Phi)_+\rangle_n$$

$$= \sum_{n=0}^{N-1}\langle k_n\frac{[\mathbf{w}_h]}{k_n}, (\hat{\Phi}_n - P\Phi)_+\rangle_n + \sum_{n=0}^{N-1}\langle k_n\frac{[\mathbf{w}_h]}{k_n}, (P\Phi - \Phi)_+\rangle_n := II_1 + II_2.$$

To estimate II_1, we use (26) to get

$$II_1 = \sum_{n=0}^{N-1}\langle k_nR_1, (\hat{\Phi}_n)_+ - P\Phi_+\rangle_n = \sum_{n=0}^{N-1}\langle R_1, k_n\hat{\Phi}_n - Pk_n\Phi_+\rangle_n$$

$$= \sum_{n=0}^{N-1}\langle R_1, k_n\hat{\Phi}_n - \int_{I_n} P\Phi(.,t)dt + \int_{I_n}\int_{t_n}^{t} P\Phi_\tau(.,\tau)d\tau dt\rangle_n$$

$$= \sum_{n=0}^{N-1}\int_{I_n}\int_{t_n}^{t}\langle R_1, P\Phi_\tau(.,\tau)\rangle_n d\tau dt$$

by using (16), (17) and Hölder inequality, we have:

$$II_1 \leq \| k_nR_1 \|_{L_2^{\psi^{-1}}(\Omega)} \| P\Phi_t \|_{L_2^{\psi}(\Omega)} \leq \| k_nR_1 \|_{L_2^{\psi^{-1}}(\Omega)} \| \Phi_t \|_{L_2^{\psi}(\Omega)} .$$

As for the II_2-terms we can write

$$II_2 = \sum_{n=0}^{N-1}\langle k_n\frac{[\mathbf{w}_h]}{k_n}, (P\Phi - \Phi)_+\rangle_n = \sum_{n=0}^{N-1}\langle \frac{\mathbf{w}_{h,+}^n - \mathbf{w}_{h,-}^n}{k_n}, (P_n - I)k_n\Phi_+\rangle_n$$

$$= \sum_{n=0}^{N-1}\langle \frac{P_n\mathbf{w}_{h,-}^n - \mathbf{w}_{h,-}^n}{k_n}, (P_n - I)(\int_{I_n}\Phi(.,t)dt - \int_{I_n}\int_{t_n}^{t}\Phi_\tau(.,\tau)d\tau dt)\rangle_n$$

$$\leq \sum_{n=0}^{N-1}\int_{I_n}\langle \frac{(P_n - I)\mathbf{w}_{h,-}^n}{k_n}, (P_n - I)\Phi(.,t)\rangle_n dt$$

$$+ \sum_{n=0}^{N-1}\int_{I_n}\int_{t_n}^{t}\langle \frac{(P_n - I)\mathbf{w}_{h,-}^n}{k_n}, (P_n - I)\Phi_\tau(.,t)d\tau dt\rangle_n$$

by using (16), (17) and Hölder inequality, we have:

$$II_2 \leq \| k_nR_2 \|_{L_2^{\psi^{-1}}(\Omega)} \| \Phi_{xx} \|_{L_2^{\psi}(\Omega)} + \| k_nR_2 \|_{L_2^{\psi^{-1}}(\Omega)} \| \Phi_t \|_{L_2^{\psi}(\Omega)} .$$

The a posteriori error estimate now follows immediately after collecting the terms and using the definition of the stability factors (24) and (25). □

8. Numerical results

At present, three numerical examples for testing Sd method are given. We carry out (7), by an AMD Opteron computer with 15 Gigabytes RAM memory with 2.2 GHz CPU. For each slab S_n, let x_i^n be a mesh, portioned into intervals $J_i^n = (x_{i-1}^n, x_i^n)$, with $h_i^n = x_i^n - x_{i-1}^n$. We define the time mesh function $k = k(t)$ by $k(t) = k_n$ for $t \in (t_n, t_{n+1})$. For $h > 0$ let T_h^n be a triangulation of the slab S_n into triangle K (cf. Figure 1.), satisfying as usual the minimum angle condition (see, e.g. [33]), and indexed by the parameter h representing the maximum diameter of the triangle $K \in T_h^n$. The triangulation of S_n may be chosen independently of that of S_{n-1}, but for the sake of simplicity it must satisfy quasi-uniformity conditions for finite element meshes [12]. To give numerical results obtained using the Sd method, we shall use finite element approximation on a space time slab with the trial function which are piecewise polynomials in space and linear in time; that is, for $(x,t) \in S_n$, we let $w_h^n(x,t) = (u_h^n(x,t), v_h^n(x,t))^T \in \mathbf{W}_h^n$ where

$$u_h^n = \sum_{i=1}^{M} \phi_i(x)(\theta_1(t)\widetilde{u_i^n} + \theta_2(t)u_i^{n+1}) \text{ and}$$

$$v_h^n = \frac{\partial u_h^n}{\partial t} = \sum_{i=1}^{M} \phi_i(x)(\theta_1'(t)\widetilde{v_i^n} + \theta_2'(t)v_i^{n+1})$$

such that $\{\phi_i(x_j) = \delta_{ij}\}$, $i,j = 0,\cdots,M$ are the spatial shape functions at node i and $\{\theta_1 = \frac{t_{n+1}-t}{k}, \theta_2 = \frac{t-t_n}{k}\}$ are the time linear interpolation functions. Moreover, we assume the nodal values of u for node i ant $(t_n)_+$ and $(t_{n+1})_+$ are denoted by $\widetilde{u_i^n}$ $(=\widetilde{v_i^n})$ and u_i^{n+1} $(= v_i^{n+1})$, respectively. Therefore, we consider the above algorithm for the following test problems.

Tabl. 1. Error $= \|w - w_h^n\|_\infty$ by Sd method at different δ.

(x,t)	$\delta = 0.15$	$\delta = 0.10$	$\delta = 0.05$	$\delta = 0.01$	$\delta = 0.005$
$(-1, 0.1)$	0.231e-6	0.212e-9	0.431e-8	0.751e-10	0.321e-9
$(0.0, 0.5)$	0.231e-5	0.761e-7	0.454e-7	0.983e-9	0.522e-10
$(1, 0.9)$	0.514e-7	0.634e-10	0.713e-10	0.761e-9	0.510e-9

Test problem 1. Streamline diffusion method is computed by given δ, $\beta = 0$, $M = 20$, $h = 0.1$, $k = 0.005$, $u(x,t) = \sin\xi(x+t)$ and $v(x,t) = \sin\zeta(x+t)$ such that we define $\xi(x) = \xi(x+2)$, $\zeta(x) = \zeta(x+2) + \pi$ and $\xi = \begin{cases} \pi/2 & x = 0 \\ x & x \neq 0 \end{cases}$
Therefore, we have the exact solution of (1) and in Table 1., we verify pointwise of the error $= \|w - w_h^n\|_\infty = \max\{|u(x,t) - u_h^n(x,t)|, |v(x,t) - v_h^n(x,t)|\}$. In this example we test how well the stability theory developed in Proposition 5 matches with computation by the stability factors that is (24) and (25). Therefore, this proposition guarantees computational stability for small time step.

Test problem 2. Streamline diffusion method is shown by given $\beta = 0$ and $F(x, t, u) = (u - 1)^2$ in Figure 2 (in the first row). The results are given after 10 time step that is $n = 1, 2, ..., 10$ and $k_n = 0.1$. In this example, we haven't the exact solution but Proposition 5 guarantees computational stability.

Test problem 3. Streamline diffusion method is shown by given $\alpha = 0$ and $F(x, t, u) = (u-1)^2$ in Figure 2 (in the second row). The results are given after 10 time step that is $n = 1, 2, ..., 10$ and $k_n = 0.1$. In this example, we haven't the exact solution but Proposition 5 guarantees computational stability.

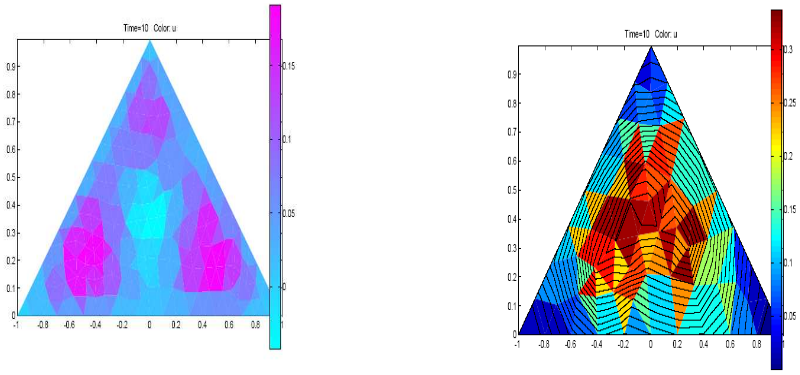

FIG. 2. The approximation solution of u for example 2 (in the first row) and example 3 (in the second row) when $\delta = 0.1$ and the stability factors $Y_e^t, Y_e^x \leq 10^{-3}$

9. Conclusion

To this end, a special nonlinear second order hyperbolic initial-boundary value problem is investigated. We use streamline diffusion method for this case of this wave equation and obtain a priori and a posteriori error estimates. A posteriori error estimate is a very powerful mathematical tool in this problem by Sd method. We try to obtain optimal bounds and the eigenvalues and eigenfunctions remains a challenge that deserves special attention and will be consideration elsewhere.

Acknowledgments

The authors express their gratitude to anonymous refere for many helpful remarks and suggestions.

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Received 11.04.2012