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Preconditioners for Solving Stochastic Boundary Integral Equations with Weakly Singular Kernels

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Abstract

A stochastic linear heat conduction problem is reduced to a special weakly singular integral equation of the second kind. The smoothness of the solution to a multidimensional weakly singular integral equation is investigated. It is also indicated that the derivatives of solutions may have singularities of certain order near the boundary of domain. The solution in the form of a multidimensional cubic spline is studied using circulant integral operators and a special mesh near the boundary with respect to all variables. Furthermore, stable numerical algorithms are given.

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1. Introduction

Stochastic linear integral equations arise in a variety of practical applications in mathematics and engineering, especially in the solution of transient heat conduction with white noise [1, 9, 11, 15, 20, 22]. Moreover, if geometrical complexities or the form of boundary conditions preclude their use, recourse must be made to finite-difference and finite-element methods by the computer. Such methods may be used to solve any conduction problems, with different levels of complexity.

The present study deals with the following equation:

$$\sigma(x) = \int_{G'} a(x, y)k(x - y)\sigma(y)dy + w(x), \tag{1}$$

where $a \in C^m(G' \times G')$, $k \in C^{m-1}(G' \setminus \{0\})$, $m \geq 1$, $G' \subseteq \mathbb{R}^d$ and w a given process defined on a probability space (or a given function from a Banach space). This equation can be generated from a stochastic linear heat conduction such as:

$$\begin{cases} \bigtriangledown^2 u - K^2 u = \overset{\bullet}{W}(x, t) \\ \frac{\partial u}{\partial n}|_S = 0, \quad x \in G_e \subseteq \mathbb{R}^d, \end{cases} \tag{2}$$

where $\overset{\bullet}{W}$ is a space-time noise and S is the boundary of G_e (exterior region to S) and G_i is the interior region, while n stands for the unit normally outwards to S, assuming S is a smooth or regular boundary and K is a constant. Suppose that μ be a σ-finite measure on $\mathbb{R}^{d-1}$, $\overline{\mu}$ the measure $d\mu \times dt$ on $\mathbb{R}^d$, and W a Gaussian additive set function of the Borel sets of $\mathbb{R}^d$ such that $W(A)$ is a Gaussian random variable of mean zero and variance $\overline{\mu}(A)$ so that $W(A)$ and $W(B)$ are independent if $A \cap B = \phi$. In this case, the $x = (x_1, x_2, \ldots, x_{d-1}) \in \mathbb{R}^{d-1}$ and $W_{xt} = W\{(x_1, \infty) \times \ldots \times (x_{d-1}, +\infty) \times (0, t)\}$ where W is not differentiable, but its derivative can exist. If we consider $\overset{\bullet}{W}$ as $\frac{\partial^d W_{x,t}}{\partial x_1 \ldots \partial x_{d-1} \partial t}$, then system (2) is very interesting when $d = 3$, since the problem embodies the solution of an infinite expanse of material containing a cavity S on which the temperature gradient is zero. Therefore, throughout the present paper "d" can be replaced by "3".

It is obvious the heat conduction problem is encapsulated by an integral equation if we assume that u can be represented as the sum of a volume potential and a single-layer potential as follows:

$$4\pi u(x) = \int_{G_e} \overset{\bullet}{W}(\xi) E(x;\xi) dV + \int_S \sigma(\xi) E(x;\xi) dS, \tag{3}$$

where $E(x, \xi) = \frac{exp(-K|x-\xi|)}{|x-\xi|}$ is a fundamental solution of the Helmholtz equation, σ is an unknown source density, with $x', \xi' \in \mathbb{R}^{d-1}$, $x = (x', t)$, $\xi = (\xi', t)$ and $|x| = (x_1^2 + ... + x_{d-1}^2 + t^2)^{\frac{1}{2}}$.

The next step is to take the normal derivative of both sides of Eq. (3), let x approach S and use the equations given in the Appendix. Then the result is a weakly singular integral equation of the second kind in $\sigma(x)$ for (1) when

$$w(x) = \int_{G_e} \overset{\bullet}{W}(\xi) \frac{\partial E(x,\xi)}{\partial n} dV,$$

and $\int_{G'} a(x, y)k(x - y)\sigma(y)dy$ is made from $\int_S \sigma(\xi)\frac{\partial E(x,\xi)}{\partial n} dS$ so that

$$G' = \{x \in \mathbb{R}^d, 0 \leq x_j < \infty, j = 1, 2, \ldots, d\}.$$

(Of course, w can be supposed to be a Wiener process defined on an appropriate probability space.)

For details of the one-dimensional Wiener-Hopf equations see Gohberg and Feldman 1974 [4]. In this paper the results are developed for multidimensional equations on a unbounded region with weakly singular kernel.

The outline of the paper is as follows. In Section 2 solution smoothness is discussed since the exact solution may not be smooth. In Section 3, we rewrite the integral equation by circulant operators as preconditioners similar to those that will be constructed afterwards. The convergence analysis of the preconditioned operators are discussed in Section 3.4. There are many complications for discretization of an integrable singularity of the kernel. We use special grids highly concentrated near the possible singularity points that eliminate most of the difficulties. In particular we seek an approximate solution in the form of a multidimensional cubic spline. In Section 4, numerical methods involving preconditioned conjugated gradient method and employing quadrature rules as numerical algorithms are also investigated. Finally concluding remarks are given in Section 5.

2. Solution Smoothness

Consider Eq. (1) where

$$a \in C^m(G' \times G'), \quad k \in C^{m-1}(G' \setminus \{0\}), \quad m \geq 1, G' \subseteq \mathbb{R}^d. \tag{4}$$

Differentiation of a weakly singular kernel increases the order of the singularity, e.g. $D_x^{\beta}|x-y|^{-v}$ $(v > 0)$ behaves as $|x-y|^{-v-|\beta|}$. This observation motivates following smoothness assumption about the kernel:

the kernel $k(x-y)a(x,y)$ is m times $(m \geq 1)$ continuously differentiable on $(G' \times G') \setminus \{x = y\}$ and there exists a real $v \in (-\infty, d)$ so that the estimate

$$|D_x^{\alpha} D_{x+y}^{\beta} k(x-y)| \leq C \begin{cases} 1 & v + |\alpha| < 0 \\ 1 + |log|x-y|| & v + |\alpha| = 0 \\ |x-y|^{-v-|\alpha|} & v + |\alpha| > 0 \end{cases},$$
$$C = \text{constant} \quad x, y \in G', \tag{5}$$

is valid for all multi-indexes $\alpha = (\alpha_1, ..., \alpha_d) \in Z_+^d$ and $\beta = (\beta_1, ..., \beta_d) \in Z_+^d$ with $|\alpha| + |\beta| \leq m$.

Here the following usual conventions are adopted:
$|\alpha| = \alpha_1 + \cdots + \alpha_d$ for $\alpha = (\alpha_1, ..., \alpha_d) \in Z_+^d$, $|x| = (x_1^2 + ... + x_d^2)^{1/2}$ for $x \in \mathbb{R}^d$ and $D_x^{\alpha} = (\frac{\partial}{\partial x_1})^{\alpha_1} ... (\frac{\partial}{\partial x_d})^{\alpha_d}$, $D_{x+y}^{\beta} = (\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1})^{\beta_1} ... (\frac{\partial}{\partial x_d} + \frac{\partial}{\partial y_d})^{\beta_d}$ denote the differential operators.

Putting $|\alpha| = |\beta| = 0$, thus $k(x-y)$ may have a weak singularity $(v < d)$. In the case $v < 0$, the kernel is bounded but its derivatives may be singular.

Let us introduce the Banach function space (as a solution space for σ)

$$E_\tau(G') = E_\tau^{\beta,m} =$$
$$\{\sigma \in C(G') \cap C^m(G') | \sum_{|\beta|\leq m} \sup_{x\in G'} \frac{|D^{|\beta|}\sigma(x)|}{|x|^{-|\beta|} + (|\tau - |x||)^{-|\beta|}} < \infty\},$$

equipped with the norm

$$\|\sigma\|_{E_\tau(G')} = \max_{x\in G'} |\sigma(x)| + \sum_{|\beta|\leq m} \sup_{x\in G'} \frac{|D^{|\beta|}\sigma(x)|}{|x|^{-|\beta|} + (|\tau - |x||)^{-|\beta|}},$$

where $0 < \tau \leq \infty$ and $x = (x_1, ..., x_d) \in G'$.

The behavior of lower derivatives of the functions $\sigma \in E_\tau^{\beta,m}$ is slightly different for non-integer and integer β.

For non-integer $|\beta|$ the inclusion $\sigma \in E_\tau(G')$ is equivalent to the condition $\sigma \in C^j(G') \cap C^m(G')$, where $j = int(m - |\beta|)$ is the integer part of $(m - |\beta|)$,

$$|D^k\sigma(x)| \leq c_k[|x|^{-|\beta|+m-k} + (|\tau - |x||)^{-|\beta|+m-k}],$$
$$k = j+1, .., m \quad , c_k = \text{constant}.$$

For integer $|\beta|$ $(1 \leq |\beta| \leq m-1)$, the inclusion $\sigma \in E^\beta$ implies that $\sigma \in C^{j-1}(G') \cap C^m(G'), \quad j = m - |\beta| \geq 1$

$$|D^j\sigma(x)| \leq c_j\big(|log|x|| + |log|\tau - |x||| + 1\big),$$

$$|D^k\sigma(x)| \leq c_k\big(|x|^{-|\beta|+m-k} + (|\tau - |x||)^{-|\beta|+m-k}\big),$$

$k = j+1, ..., m \quad c_j, c_k = constant$. The smoothness of the solution to (1) is described by the following theorem (see also [5, 7, 10]).

Theorem 1. *Let conditions (4) and (5) be satisfied and let* $w \in E_\tau(G')$. *If equation (1) has an integrable solution* σ, *then* $\sigma \in E_\tau(G')$.

Proof: We have to prove that any solution $\sigma \in L_\infty(G')$ of Eq. (1) belongs to $E_\tau(G')$. The idea that we follow is to replace Eq. (1) by an equation on a small open subset $\Omega \in G'$ so that $(I - A_\Omega)[\ ]$ is invertible in both spaces $L_\infty(\Omega)$ and $E_\tau(G')$ (where $(I - A_\Omega)[\sigma] = \sigma - \int_\Omega a(x,y)k(x-y)\sigma(y)dy = w$). If the operator A_Ω maps $E_\tau(\Omega)$ into $E_\tau(\Omega)$ and for any $\sigma \in E_\tau(\Omega)$ then it is clear that

$$||A_\Omega\sigma||_{L_\infty(\Omega)} \leq c_0||\sigma||_{L_\infty(\Omega)}. \tag{1*}$$

Therefore $(I - A_\Omega)[\,]$ is invertible in $L_\infty(G')$ [10]. Let us prove the invertibility in $E_\tau(\Omega)$. Introduce in $E_\tau(\Omega)$ a provisional new norm $||\sigma||'_{E_\tau(\Omega)} = 4||\sigma||_{L_\infty(\Omega)} + ||\sigma||_{E_\tau(\Omega)}$ which is equivalent with the old one: $||\sigma||_{E_\tau(\Omega)} \leq ||\sigma||'_{E_\tau(\Omega)} \leq 4||\sigma||_{E_\tau(\Omega)}$.

Using (1*) we find

$$||A_\Omega\sigma||' = 4||A_\Omega\sigma||_{L_{\infty(\Omega)}} + ||A_\Omega\sigma||_{E_\tau(\Omega)} \leq$$

$$\leq 4c_0||\sigma|| + \max_{x\in\Omega}|A_\Omega\sigma| + \sum_{|\beta|\leq m}\sup_{x\in\Omega}\frac{|D^{|\beta|}A_\Omega\sigma|}{|x|^{-|\beta|} + (|\tau - |x||)^{-|\beta|}} \leq$$

$$\leq (4c_0 + 2)||\sigma||_{L_\infty(\Omega)} + \sum_{|\beta|\leq m}\sup_{x\in\Omega}\frac{|D^{|\beta|}\sigma|}{|x|^{-|\beta|} + (|\tau - |x||)^{-|\beta|}} \leq$$

$$\leq c_1||\sigma||'_{E_\tau(\Omega)},$$

where c_0 and c_1 are constant.

A consequence is that $(I - A_\Omega)[\,]$ is invertible in $L_\infty(\Omega)$ and $E_\tau(\Omega)$. Let $\sigma_0 \in L_\infty(\Omega)$ be a solution to Eq. (1). We have to prove that $\sigma_0 \in E_\tau(G')$. Notice that the restriction of σ_0 to Ω satisfies the equation

$$\sigma(x) = \int_\Omega k(x-y)a(x,y)\sigma(y)dy + w_\Omega(x), \quad x \in \Omega \tag{2*}$$

where

$$w_\Omega(x) = w(x) + \int_{G\setminus\Omega} k(x-y)a(x,y)\sigma_0(y)dy, \quad x \in \Omega.$$

An important observation is that $w_\Omega \in E_\tau(\Omega)$. On the other hand, we see that $\int_{G\setminus\Omega} k(x-y)a(x,y)\sigma_0(y)dy$ belong to $E_\tau(\Omega)$ and

$$||w_\Omega||_{E_\tau(\Omega)} \leq ||w||_{E_\tau(G')} + c||\sigma_0||_{L_\infty(G')} \quad c = \text{constant}.$$

Thus, Eq. (2*) is uniquely solvable in $E_\tau(\Omega)$. It is clear Eq. (2*) is uniquely solvable in $L_\infty(\Omega)$ and we know that the solution (it is σ_0) is the restriction to Ω of the solution of (1) under consideration. Since $E_\tau \subset L^\infty(\Omega)$, we obtain $\sigma \in E_\tau(G')$.

The theorem implies that the l-th derivative of a solution of Eq. (1) for $|x| \to 0$ and $|x| \to \tau$ may have the same singularities that $D^{(l-1)}k(x)$ has for $|x| \to 0$.

Remark 1. Without loss of generality of problem, we assume the integral operator to be self-adjoint and the operator is also positive definite.

3. Circulant Integral Operators and Special Grids

Regarding the solution smoothness two ideas for the rate of convergence can be stated:

a. The use of circulant preconditioned operators that could increase the rate of convergence of the conjugate gradient method [6,21].
b. The use of special grids highly concentrated near the possible singularity points can ensure that the rate of convergence of a collocation method is optimal in terms of the uniform norm [7, 10].

In this section, the approximate solution is studied by combining these two ideas, i.e. circulant integral operators and special grids for discretization of circulant operators.

3.1. Preliminary

One way of solving (1) is to use the projection method [4,10], where the solution $\sigma(x)$ of (1) is approximated by the solution $\sigma_\tau(x)$ of the equation

$$\sigma_\tau(x) = \int_{G''} a(x, y)k(x - y)\sigma_\tau(y)dy + w(x), \tag{6}$$

where $G'' = \{x \in G'|0 \leq x_j < \tau, j = 1, 2, ..., d\}$[4-6]. It is clear that $\lim_{\tau\to\infty} \|\sigma_\tau - \sigma\|_{L_p(G'')} = 0$ for $1 \leq p < \infty$. If we define

$$A_\tau[\ ] = \begin{cases} \int_{G''} a(x, y)k(x - y)[\ \ ]dy, & x \in G'' \setminus \{x = y\} \\ 0 & x \notin G'' \setminus \{x = y\}, \end{cases}$$

then $A_\tau[\ ]$ is a compact operator and $A_\tau : L_p(G'' \setminus \{x = y\}) \to E_\tau^{\beta,m}$. The spectrum of the operator $I - A_\tau$ is clustered around 1 because that is a compact operator (where I is the identity operator). Therefore solving (6) by iterative methods such as the Conjugate Gradient (CG) method will be less expensive than using direct methods [10, 13, 14]. One standard way for speeding up the convergence rate of the CG method is to apply a preconditioner. As $\tau \to \infty$, the spectrum of the operator A_τ becomes dense in the spectrum of the operator defined by (1). Hence the convergence rate of the CG will deteriorate [6]. (See the numerical results in Section 4 and the following remark.)

Remark 2. It is clear that the iterations $\sigma_k, k = 1, 2, ...$ of the CG, in general, converge to the solution σ_τ of (6) with a linear rate of convergence $||\sigma_\tau - \sigma_k||_{E_\tau(G')} \leq c(\frac{l-1}{l+1})^k$. Where c is positive constant and the $\frac{l-1}{l+1}$ depends on the condition number of the operator A_τ (i.e. $l = l(A_\tau) = \sqrt{||A_\tau||\ ||A_\tau^{-1}||}$)[6,13].

Thus instead of solving (6), we propose two ways based on the preconditioner idea. The first way is as follows:

$$(I + C_\tau)^{-1}(I - A_\tau)\sigma_\tau = (I + C_\tau)^{-1}w(x), \tag{7}$$

where the operator C_τ is a preconditioner for the operator A_τ. The second way which is based on Gohberg et al. in [6], as follows:

$$(I + C_\tau)\sigma_\tau = w(x), \tag{8}$$

where the circulant integral operator is a good preconditioner of $C_\tau[\ ]$ in the sense that it is close to $A_\tau[\ ]$ in some norm.

Remark 3. We remark that the following bound for the convergence of the PCG algorithm that may be applied to solve $C^{-1}(I - A_\tau)\sigma = C^{-1}w$:

$$||\sigma_\tau - \sigma_k||_{E_\tau(G')} \leq c_1\left(\frac{\epsilon}{1 + \sqrt{1 - \epsilon^2}}\right)^k, \quad c_1 = \text{constant},$$

if the spectral elements of $C^{\frac{-1}{2}}(I - A_\tau)C^{\frac{-1}{2}}$ are contained in $(1 - \epsilon, 1 + \epsilon)$ with $\epsilon < 1$ (see [6,13]).

Usually the Eq. (8) is rather easier to solve than Eq. (7). Let us introduce

$$\mathcal{G} = \left\{C_\tau | C_\tau[\ \ ] = \int_{G''} h_\tau(x - y)[\ \ ]dy, \ \ x, y \in G'' \setminus \{x = y\}\right\},$$

as a class of preconditioners (where h_τ is the periodic function in $G'' \setminus \{x = y\}$ and conditions (4),(5) are satisfied for it). Choosing $C_\tau[\ ]$ is based on two standard ways:

(i) to minimize $\|A_\tau - C_\tau\|^2$

(ii) to minimize $\|I - (I + C_\tau)^{-1}(I - A_\tau)\|^2$

on $\mathcal{G}$ with the Hilbert-Schmidt norm, where the first circulant integral is called the optimal circulant integral and the second is the called the super-optimal circulant integral. In [6] the authors assumed that in (6), $a(x, y) = 1$ and the kernel does not have a singularity but here it is supposed that $a(x, y) \neq 1$ and the kernel function has a singularity.

3.2. Construction of the Optimal Circulant Integral

If the first minimizer is called the optimal preconditioner and is denoted by $p(A_\tau)$, then it is possible to construct the optimal circulant integral preconditioner $p(A_\tau)$ for integral operators A_τ. The preconditioner $p(A_\tau)$ is defined to be the circulant integral operator that minimizes the Hilbert-Schmidt norm

$$\|A_\tau - H_\tau\|^2 = \int_{G''}\int_{G''} (a(x,y)k(x-y) - h_\tau(x-y))(\overline{a}(x,y)\overline{k}(x-y) - \overline{h}_\tau(x-y))dydx, \tag{9}$$

over all circulant integral operators C_τ or $\mathcal{G}$.

The expression of the kernel function of $p(A_\tau)$ is presented first. Since $a, k \in L_p(G'' \setminus \{x = y\})$, by using Fourier expansions we have

$$a(x,y)k(x-y) = \sum_{m,n=-\infty}^{\infty} v_{m,n} u_m(x)\overline{u}_n(y) \ , \quad x \neq y, \tag{10}$$

where u_m is an eigenfunction of the operator C_τ with the following eigenvalue:

$$\lambda_m = \sqrt{\tau}(h_\tau, u_m) = \sqrt{\tau}\int_{G''} h_\tau(x)\overline{u}_m(x)dx, \quad m \in Z, \tag{11}$$

and

$$v_{m,n} = \int_{G''}\int_{G''} a(x,y)k(x-y)u_n(y)\overline{u}_m(x)dxdy = (A_\tau u_n, u_m)_\tau \ , \quad m, n \in Z, \tag{12}$$

By means of the Fourier expansion, we can write

$$h_\tau(x-y) = \sum_{m=-\infty}^{\infty} \lambda_m u_m(x)\overline{u}_m(y), \qquad x, y \in G'' \setminus \{x = y\}.$$

Combining this equation with (10) and using the orthogonality of u_n such as $u_m(x) = \frac{1}{\sqrt{\tau}}e^{2\pi i m x/\tau}$, $m \in Z$, we can express the distance (11) as

$$\|A_\tau - C_\tau\|^2 = \sum_{m=-\infty}^{+\infty} |v_{m,m} - \lambda_m|^2 + \sum_{\substack{m,n=-\infty \\ m\neq n}}^{\infty} |v_{m,n}|^2.$$

Clearly, the expression becomes minimal if and only if $\lambda_m = v_{m,m} = (A_\tau u_m, u_m)_\tau$ for all integers m.

Remark 4. Here, we consider that the given integral operator in (6) is close to a convolution-type integral operator in (8). On the other hand, we know that there is an intimate relation between convolution integral equations and Toeplitz operators (or semi-infinite Toeplitz) see [4]. Therefore, we will prove in the Section 3.4 that the given integral operator is close to a convolution-type integral operator. Thus the preconditioned system (8) have a convergence rate of superlinear with preconditioned conjugate gradient method (see Remark 3). In other words, let $\mathbf{A}$, $\mathbf{C}$ be as the matrices that has been discritized from the operators of (6) and (8) respectively. Then $\mathbf{C}$ for $\mathbf{A}$ can be viewed as an approximation to $\mathbf{A}$ and can be used to obtain the following iterative method for solving the system (6) or $\mathbf{A}x = w$. The convergence of this iteration (or an acceleration of it by conjugate gradient or other methods) depends on the spectrum of $\mathbf{C}^{-1}(\mathbf{C} - \mathbf{A})$: the smaller $\boldsymbol{\lambda}(\mathbf{C}^{-1}(\mathbf{C} - \mathbf{A}))$, the faster the convergence. It can be easily shown that, if $\mathbf{B} = \mathbf{C} - \mathbf{A}$, then

$$\boldsymbol{\lambda}(\mathbf{C}^{-1}\mathbf{B}) \leq ||\mathbf{C}^{-1}\mathbf{B}|| = ||(\mathbf{I} + \mathbf{A^{-1}B})^{\mathbf{-1}}\mathbf{A^{-1}B}|| \leq \frac{||\mathbf{A}^{-1}\mathbf{B}||}{1 - ||\mathbf{A}^{-1}\mathbf{B}||},$$

provided $||\mathbf{A}^{-1}\mathbf{B}|| < 1$ in some consistent matrix norm. Since the upper bound is a strictly increasing function of $||\mathbf{A}^{-1}\mathbf{B}||$, it is natural to chose $\mathbf{C}$ such that $||\mathbf{A}^{-1}\mathbf{B}||$ is minimized. In general, the convergence rate of the conjugate gradient iteration depends on the spectrum and this problem is studied in detail in more recent papers (for example see [6,13,21]). Therefore, comparing the convergence rate of (8) with (6) is clear.

On the other hand, we can have a remark similar to this for (7), (6) in the end of Section 3.3 that we dispense with it.

3.3. Construction of the Super-Optimal Circulant Integral

This section is an extension of an idea presented by Tyrtyshikov and Strela [21]. We consider another type of circulant integral operator which is obtained by minimizing the Hilbert-Schmidt norm

$$\|I - (I + C_\tau)^{-1}(I - A_\tau)\|^2, \tag{13}$$

over all circulant integral operators of $\mathcal{G}$ such that the operator $(I+C_\tau)^{-1}$ exists. Let $C_\tau[\ ]$ be a circulant integral operator with kernel function and eigenvalues λ_m given in (11). If $(I + C_\tau)[\ ]$ is invertible, then $1 + \lambda_n \neq 0$ for all integers n. Moreover, since h_τ is in $L_2(G'' \setminus \{x = y\})$ and $\sum_{n=-\infty}^{\infty}(\lambda_n)^2 < \infty$, therefore $|1+\lambda_n| \geq 1/2$ for all $|n|$ that are sufficiently large. On the other hand $(I + C_\tau)^{-1} = I - K_\tau$, where K_τ is also a circulant integral operator with kernel function

$$k_\tau(x - y) = \sum_{n=-\infty}^{+\infty} \frac{-\lambda_n}{1 + \lambda_n} u_n(x)\overline{u}_n(y), \qquad x \neq y. \tag{14}$$

Since, the kernel function of $C_\tau K_\tau$ at the point $(t-s)$ is given by

$$\int_{G''}\sum_{m=-\infty}^{+\infty}\lambda_m u_m(t)\overline{u}_m(y')\sum_{n=-\infty}^{+\infty}\frac{-\lambda_n}{1+\lambda_n}u_n(y')\overline{u}_n(s)dy' = \sum_{n=-\infty}^{\infty}\frac{-\lambda_n^2}{1+\lambda_n}u_n(t)\overline{u}_n(s),$$

the function k_τ is a τ-periodic function and it is straightforward to check the kernel function

$$C_\tau - K_\tau - C_\tau K_\tau = (I+C_\tau)(I-K_\tau) - I.$$

The problem of minimizing the norm (13) becomes the problem of minimizing $\|I-(I-K_\tau)(I-A_\tau)\|$ over all circulant integral operators K_τ. We assume the $v_{n,m}=(A_\tau u_n, u_m)$ and $\xi_m=\sum_{n=-\infty}^{+\infty}|v_{n,m}|^2$. Then, $\|I-(I-K_\tau)(I-A_\tau)\| = \|K_\tau + A_\tau - K_\tau A_\tau\|$ and, by using (14), the kernel function of $K_\tau - K_\tau A_\tau + A_\tau$ at point (x, y) is given by

$$\sum_{m,n=-\infty}^{\infty}\left\{-\frac{\delta_{m,n}\lambda_m}{1+\lambda_m}-\frac{\lambda_m v_{m,n}}{1+\lambda_m}+v_{m,n}\right\}u_m(x)\overline{u}_n(y) = \sum_{m,n=-\infty}^{\infty}\left(\frac{-\delta_{m,n}\lambda_m+v_{m,n}}{1+\lambda_m}\right)u_m(x)\overline{u}_n(y),$$

where $\delta_{m,n}$ denotes the Kronecker symbol. By the definition of the Hilbert-Schmidt norm,

$$\|I-(I+C_\tau)^{-1}(I-A_\tau)\|^2 = \sum_{m,n=-\infty}^{+\infty}\left|\frac{\delta_{m,n}\lambda_m - v_{m,n}}{1+\lambda_m}\right|^2,$$

it is clear that the above expression is minimized if and only if the term

$$\frac{|\lambda_m|^2-\lambda_m\overline{v}_{m,m}-\lambda_m v_{m,m}+\xi_m}{|1+\lambda_m|^2},$$

is minimized for all integers m. However, by differentiating this quotient with respect to the real and imaginary parts of λ_m, we see that minimum will be obtained if we set $\lambda_m = \frac{\xi_m+v_{m,m}}{1+\overline{v}_{m,m}}$ for all m. Hence, the kernel function of super-optimal circulant $C_\tau[\ ]$ is given by:

$$c_\tau(x-y) = \sum_{m=-\infty}^{+\infty}\left(\frac{\xi_m+v_{m,m}}{1+\overline{v}_{m,m}}\right)u_m(x)\overline{u}_m(y), \qquad x\neq y.$$

Moreover, since

$$1+\lambda_m = \frac{\sum_{n=-\infty}^{+\infty} |v_{m,n}|^2 + |1+\overline{v}_{m,m}|^2}{1+\overline{v}_{m,m}} \neq 0,$$

we can see that $(I + C_\tau)[\ ]$ is invertible.

According to Remarks 4, 3 and Eqs. (6), (7), we can say that the rates of convergence for the optimal and super-optimal preconditioners are fast. For proving our claim, we bring the following section.

3.4. Convergence analysis

In this section, we analyze the convergence rate of the optimal and super-optimal circulant integrals as preconditioners for solving the integral equation of the second kind. Gohberg et al. in [6] had studied the convergence rate of the one-dimensional Wiener-Hopf equation (see Theorem 3.1 of [6]). We extend the theorems by Gohberg et al. in the following theorem.

Theorem 2. *Let A be a self-adjoint integral operator with the kernel function $a(x, y)k(x-y) = k_1(x-y) + a_1(x, y)$ where $k_1(.) \in C^{m-1}(G' \setminus \{0\}) \cap L_2(G')$ and $a_1(x, y)$ satisfies $\int_{G'}\int_{G'} |a_1(x, y)|^2 dxdy \leq M < \infty$, $M = constant$. Let A_τ be the defined operator in section 3.1 and C_τ be the optimal preconditioner for A_τ. Then for each $\epsilon > 0$, there is a $\rho \in \mathbb{N}$ and a $\tau^\star > 0$ with the following properties:*

(i) for each $\tau \geq \tau^\star$ there exists a decomposition $A_\tau - C_\tau = R_\tau + E_\tau$ where R_τ and E_τ are self-adjoint operators.
(ii) The spectrum of $(I + C_\tau)^{\frac{-1}{2}}(I - A_\tau)(I + C_\tau)^{\frac{-1}{2}}$ has at most ρ eigenvalues outside the interval $(1-\epsilon, 1+\epsilon)$ whenever $\tau \geq \tau^\star$.

Proof: Let A_1, A_2 be self-adjoint operators with kernels $k_1(x-y)$ and $a_1(x, y)$, respectively. Since A_1 has a convolution kernel, in this case the convergence analysis for optimal circulant integral preconditioners has been studied by Gohberg et *al.*, (see [6]). On the other hand, if $\int_{G'}\int_{G'} |a_1(x, y)|^2 dxdy \leq M < \infty$ for some constant M, then for each given $\epsilon > 0$, at most $\frac{M}{\epsilon^2}$ eigenvalues of $(A_2)_\tau$ outside the interval $(-\epsilon, \epsilon)$, (see chapter 7 of [8]). Hence, if $A[\ ] = (A_1 + A_2)[\ ]$ then the theorem is valid for the optimal circulant operator as a preconditioner (also, see Remark 3).

This theorem basically states that the spectrum of the preconditioned operator is clustered around 1. Hence, if $I - A_\tau$ is preconditioned by $I + C_\tau$, we expect fast convergence. We now give the following lemma for the super-optimal preconditioner. Finally, Theorem 2 will hold for the super-optimal circulant as a preconditioner.

Lemma. *Let all of the assumptions of Theorem 2 satisfy. Moreover, we denote* O_τ *and* S_τ *as the optimal and super-optimal preconditioners, respectively. Then we have:*
(a) $\lim_{\tau\to\infty} ||O_\tau - S_\tau||_2 = 0$,
(b) Theorem 2 is valid for the super-optimal preconditioner.

Proof (a): It is clear that for each $\epsilon_1, \epsilon_2 > 0$ and by using the results of sections 3.2 and 3.3, we have:

$$0 \le \lim_{\tau\to\infty} ||O_\tau - S_\tau||_2 \le \lim_{\tau\to\infty} ||A_\tau - O_\tau||_2 + \lim_{\tau\to\infty} ||A_\tau - S_\tau||_2 \le \epsilon_1 + \epsilon_2 \le 2\epsilon_1,$$

if $\epsilon_1 = \epsilon_2$.

Proof (b): We can state; there is a $\rho \in \mathbb{N}$ and a $\tau^\star > 0$ with the following properties:
(i) for each $\tau \ge \tau_1^\star$ there exists a decomposition $A_\tau - O_\tau = R_\tau + E_\tau$ where R_τ and E_τ are self-adjoint operators.
(ii) for each $\tau \ge \tau_1^\star$ there exists a decomposition $A_\tau - S_\tau = K_\tau + H_\tau$ where K_τ and H_τ are self-adjoint operators.
Proof (a) shows that (i) and (ii) are equivalent.

3.5. Discretization of Operator Equations

Let us consider the linear systems generated by discretizing (6), (7) and (8). For this we introduce a special class of points $\Delta \equiv \Delta^{r,n}$ so that these points $x^0, x^1, ..., x^{2n} \in G''$, $n \ge 2$, are determined by

$$x_i^k = (\frac{\tau}{2})(\frac{i+k-1}{n})^r, \quad k = 0, 1, ..., n \quad i = 1, 2, ..., d \ , \ x_i^{k+n} = \tau - x_i^{n-k}$$

where $x^k = (x_1^k, x_2^k ..., x_d^k)$. We can construct a special nonuniform triangulation mesh by *automatic mesh generation* from points $\Delta^{r,n}$. Let $\Omega_0 = \{T_0, T_1, ..., T_{2n-1}\}$ be a triangulation of G''. Our grid will consist of the vertices $y^{j,0}, y^{j,1}, y^{j,2}$ from $\Delta^{r,n}$ on each triangle T_j.

Here, the real number $r \ge 1$ characterizes the non-uniformity of Ω_0 and will be discussed later in numerical examples. (Note that for $r = 1$ the selection is uniform, whereas for $r > 1$ the points are concentrated near the boundary.)

Let $f \in C^3(G'')$ and $S_{\Delta^{r,n}}$ be the piecewise cubic polynomial interpolating $f(x)$ over a triangular grid Ω_0. Then, according to the principles given in [16, 17],

$$||f - S_{\Delta^{r,n}}|| \le cMh^3, \quad c = \text{constant}. \tag{15}$$

where $M = \max_{|i|=3}\{||D^{|i|}f||\}$, $i = (i_1, i_2, ..., i_d)$, $0 \leq i_1, i_2, ..., i_d \leq 3$, $h = \max_{i=1}^{5} |x - x^i|$ and x belongs to the interior of T_j.

Let us construct an approximate solution $\sigma_{\tau,n}(x)$ (such as $S_{\Delta^{r,n}}$) to the solution of the integral equations (6),(7) or (8). Thus on each triangle T_j of Ω_0, we construct a cubic polynomial to σ and a linear system will be generated by interpolating functions and quadrature rules (see Section 4).

In Theorem 3, an approximation property of splines is given in terms of the L_p norm. Moreover, it is proved [16,17] that under the hypotheses of Theorem 3 the approximation $\sigma_{\tau,n}$ is uniquely determined.

Theorem 3. *Let $f \in E_\tau^{\beta,3}$, $0 < |\beta| < 3$,also conditions (4),(5) be satisfied from $m = 3$, and the corresponding homogeneous equation have only the zero solution. If $r = \frac{\eta}{3-|\beta|}$ and $3 - |\beta| \leq \eta \leq 3$, then the approximation $\sigma_{\tau,n}$ obtained by the collocation method for (8) satisfies the estimate*

$$\max_{0\leq k\leq 2n} ||\sigma_{\tau,n}(x^k) - \sigma_\tau(x^k)||_{L_\infty} \leq C\epsilon_n, \quad C = \text{constant}. \tag{16}$$

Proof: Equation (8) can be considered as the operator equation $\sigma_\tau = C_\tau\sigma_\tau + w$ in Banach space $E_\tau^{\beta,3}$. The collocation can be rewritten in equivalent form $\sigma_{\tau,n} = P_nC_\tau\sigma_{\tau,n} + P_nw$, where P_n is a projection which takes any function σ_τ into its piecewise interpolation. By (15) when $P_n \to 1$ in the strong sense as $n \to \infty$ and norms $||P_n||$ are uniformly bounded. Obviously, $||\sigma_{\tau,n}(x^k) - \sigma_\tau(x^k)||_{L_\infty} \leq ||\sigma_{\tau,n} - P_n\sigma_\tau||_{L_\infty}$. Furthermore, it can easily be checked that $\sigma_{\tau,n} - P_n\sigma_\tau = (I - P_nC_\tau)^{-1}P_nC_\tau(P_n\sigma_\tau - \sigma)$, hence it follows that $||\sigma_{\tau,n} - P_n\sigma_\tau||_{L_\infty} \leq c_1||C_\tau(\sigma_\tau - P_n\sigma_\tau)||_{L_\infty}$. For $|\beta| < 3$ the operator C_τ is bounded as an operator from L_1 into L_∞. So, (16) is obtained by $||\sigma_{\tau,n} - P_n\sigma_\tau||_{L_\infty} \leq c_1||C_\tau||_{L_1\to L_\infty}||\sigma_\tau - P_n\sigma_\tau||_{L_1}$ where $c_1 = constant$. Similarly, the above theorem is valid for (6) and (7).

Obviously, if $\eta \to 3$ then the magnitude of the error ϵ_n is decreased.

4. Numerical Experiments

After discretizing the operator equations by a special nonuniform grid $\Delta^{r,n}$, we consider the algorithms called Alg.WR, Alg.SR, Alg.OR, Alg.WS, Alg.SS and Alg.OS. Here letter R denotes that the integrals are computed by the rectangular quadrature rule R_m and m is the number of integration points, while the second letter S denotes that the integrals are computed by the Simpson's $\frac{1}{3}$ quadrature rule S_m (m is the number of integration points). Consequently, every one of Eqs. (6), (7) and (8) is replaced by a system of linear equations. For solving this linear system,the CG can be used. Hence, the algorithm Alg.WR and Alg.WS are based on the CG algorithm without a preconditioner that are generated by (6), while algorithms Alg.OR, Alg.OS, Alg.SR and Alg.SS denote that the preconditioners are obtained by Optimal circulant integral (letter O) and Super-optimal circulant integral (the first letter S). In particular these last four algorithms are based on the preconditioned CG (PCG).

Remark 5. In the examples, the zero vector is used as our initial guess and the stopping criterion is $\|r_k\|/\|r_0\| < 10^{-7}$ where r_k is the residual vector of the CG and PCG methods after k iterations.

Remark 6. On the right hand side of equations (6), (7) and (8) the function $w(x)$ may be a stochastic integral. So, it can be generated as a random vector w_n by the collocation points in Section 3.4 and [1, 3, 12, 18, 19, 22]. In the examples, the performance of linear systems with normally distributed random variables as white noise is first tested, in which $E(\Delta w_{n,i}) = 0$ and $E((\Delta w_{n,i})^2) = \xi_i$ for each iteration (where $\Delta w_{n,i} = w_{n,i} - w_{n,i-1}$).
At present, two numerical examples for testing the convergence performance and stability of the algorithms are given.

Example 1. In the singular integral equation of the second kind

$$\sigma(x) = \int_G K(x, y)\sigma(y)dy + w(x), \tag{17}$$

the kernel $K(x, y) = a(x, y)log(|x - y|)$, where $x, y \in \mathbb{R}^2$, satisfies (5) with $v = 0$ if $a(x, y)$ is m times continuously differentiable on $(G \times G) \setminus \{x = y\}$. The numerical solutions are constructed by some approximate solution $\sigma_{\tau,n}$ to the integral equation (17) in the form of a two dimensional cubic spline on every triangulation of $\Delta^{r,n}$. The dominant G is a part of circle and its boundary S is $\rho(\varphi) = (b\cos(\varphi), b\sin(\varphi))$ where $0 \leq b \leq \infty$ and $0 \leq \varphi \leq \pi/4$. Tables 1 and 2 show the numbers of iterations required for the convergence performance of the algorithms when $a(x, y) = \sin(|x + y|)$. Let $h = \tau/n$ and the linear system be generated by $\tau = b$, $m = n$ and $r = 3/2$. Moreover, the random variable is Gaussian centered with variance 10^{-5} and covariance 10^{-6}, so $w(x)$ can easily be simulated by Remark 6. A proper triangulation for the region of integration G is constructed by " automatic mesh generation" from $\Delta^{r,n}$ see [16, 17].

In Tables 1 and 2, we see that the numbers of iterations are increasing rapidly with increasing τ for non-preconditioned systems. This indicates that Eq. (6) is less well-conditioned as $\tau \to \infty$. Also, we see that the iterations performance of optimal circulant operators are smaller than super-optimal circulant operators. The reasons for these observations are stated in Remarks 2, 3 and the fast convergence for the PCG in Section 3.4. We also plot the iteration counts for different non-uniformity characteristics of the grid in Figs. 1 and 2. From these figures it appears that the rate of convergence depends on r and only in an interval iteration is the count cheap. Assume $\tau = 25$, $n = 50$ and random vector w_n is simulated by

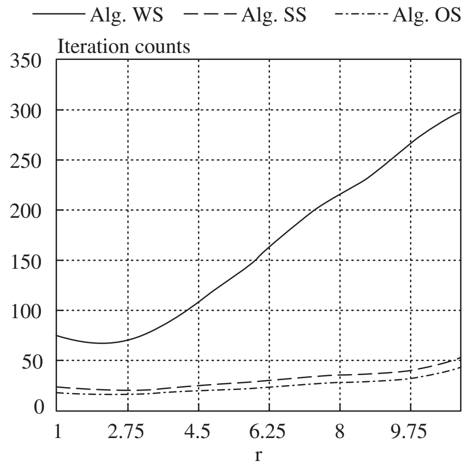

Figure 1. Different nonuniformity characteristics of the grid

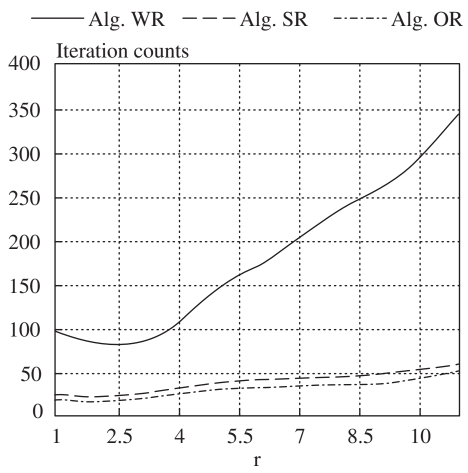

Figure 2. Different nonuniformity characteristics of the grid

variance 10^{-5} and Covariance 10^{-6}. Here, we need to a general investigation for finding an optimal real number r based on Theorem 3. For stability of perturbations of the algorithms, we plot the different variances in Fig. 3. The y-axis of Fig. 3 is the natural logarithm of the absolute value of the variance w_n. From Fig. 3, we observe that Eq. (8) and related algorithms are the stables. Here for Fig. 3, we have assumed $\tau = 25$, $n = 50$ and $r = 3/2$. Moreover, we give Table 3 regarding the relation between the number of iterations and the logarithm of condition number of the matrices (when $\tau = 25$, $r = 3/2$, random vector w_n is simulated by variance 10^{-5} and covariance 10^{-6}).

Example 2. We assume in (17), $K(x, y) = \frac{1}{4\pi} a(x) exp(-l(x, y))|x-y|^{-2}$, where

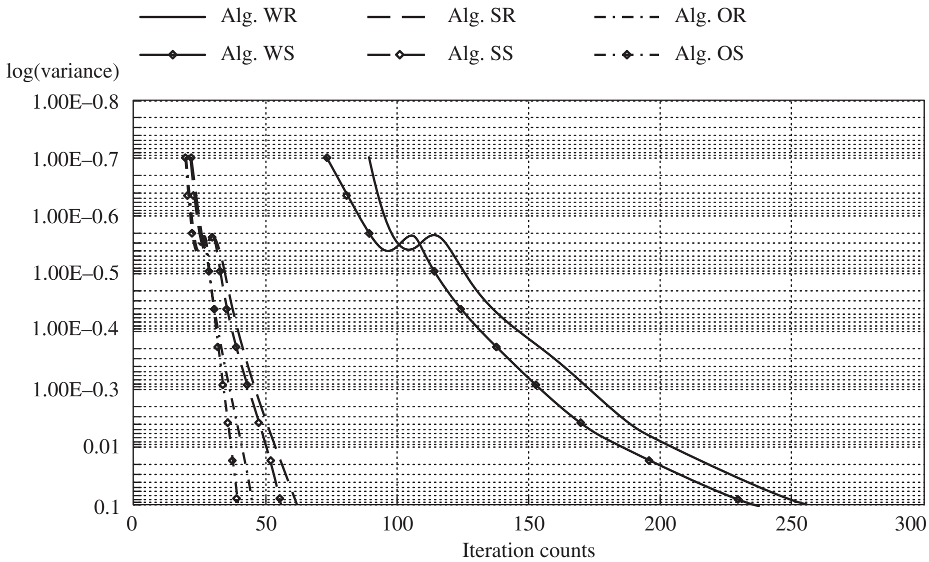

Figure 3. Different variances

Table 1. Iteration counts for example 1 with different preconditioners and the Rectangular rule (* means $>$ 1000 iterations)

	$\tau = 25$			$\tau = 75$			$\tau = 225$		
n	Alg.WR	Alg.SR	Alg.OR	Alg.WR	Alg.SR	Alg.OR	Alg.WR	Alg.SR	Alg.OR
40	87	21	19	145	24	19	293	22	20
80	136	32	27	307	33	28	631	36	27
120	183	39	38	415	44	38	937	51	39
160	632	47	44	*	51	44	*	49	45
200	*	63	57	*	69	58	*	72	57

Table 2. Iteration counts for Example 1 with different preconditioners and the Simpson rule (* means $>$ 1000 iterations)

	$\tau = 25$			$\tau = 75$			$\tau = 225$		
n	Alg.WS	Alg.SS	Alg.OS	Alg.WS	Alg.SS	Alg.OS	Alg.WS	Alg.SS	Alg.OS
40	69	20	19	116	22	19	231	23	21
80	119	29	24	254	31	25	511	32	24
120	153	35	34	345	36	33	786	35	34
160	515	42	40	*	42	40	*	44	40
200	*	59	53	*	61	52	*	60	54

$l(x, y) = |x + y|$, $x, y \in G \subset \mathbb{R}^3$ be a part of sphere in $\mathbb{R}^3$ and its boundary S is $\rho(\theta, \varphi) = (b\cos(\theta)\sin(\varphi), b\sin(\theta)\sin(\varphi), b\cos(\varphi))$ where $0 \le \theta \le \pi/2$, $0 \le$

Table 3. Number of iterations and the logarithm of condition number for Example 1 with different algorithms

n × n	Alg.	Iteration	$\log_2$(Condition number)	Alg.	Iteration	$\log_2$(Condition number)
10 × 10		17	2.3×10^1		14	1.3×10^1
20 × 20	WR	26	0.6×10^2	WS	23	0.2×10^2
30 × 30		42	8.6×10^1		39	3.9×10^1
10 × 10		10	3.6×10^{-3}		6	0.1×10^{-3}
20 × 20	SR	12	4.7×10^{-2}	SS	11	1.4×10^{-2}
30 × 30		16	7.9×10^{-2}		14	2.1×10^{-2}
10 × 10		8	3.9×10^{-3}		6	2.5×10^{-3}
20 × 20	OR	11	2.3×10^{-2}	OS	9	1.4×10^{-2}
30 × 30		13	2.7×10^{-2}		10	2.2×10^{-2}

$\varphi \leq \pi/4$, $0 < b \leq \infty$. Condition (5) is fulfilled for $v = 2$ if $a(x)$ is m times continuously differentiable and bounded on G. From testing this example, Tables 4–6 and Figs. 4–6 give similar performance. When $a(x) = exp(-|x|)$ and taking into consideration the similar assumptions for tables and figures of Example 1. Numerical results for Example 2 confirm the conclusions drawn from Example 1. Questions related to the use of an optimal real number for a nonuniform mesh will be investigated elsewhere.

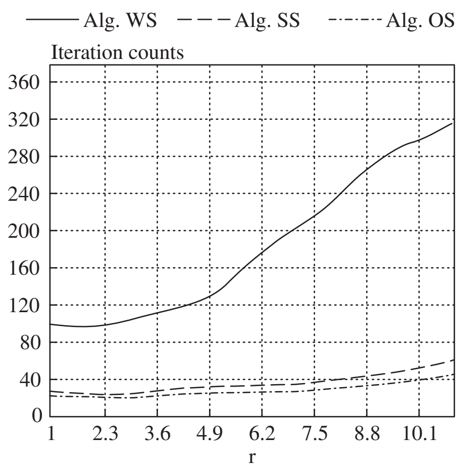

Figure 4. Different nonuniformity characteristics of the grid

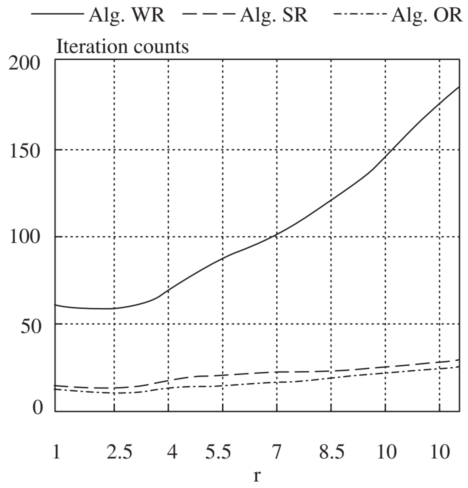

Figure 5. Different nonuniformity characteristics of the grid

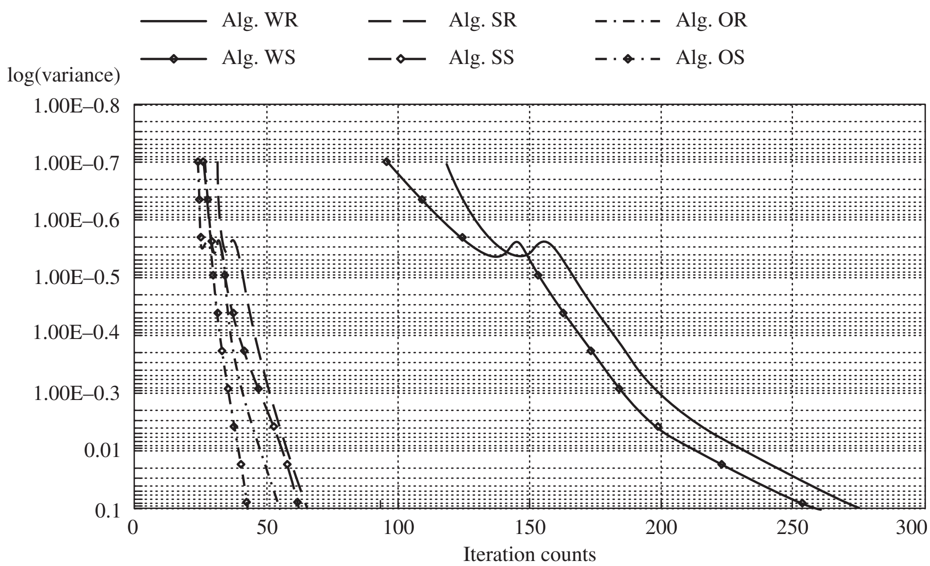

Figure 6. Different variances

5. Concluding Remarks

We remark that the accuracy of the computed solution depends on the
- determination of the special nonuniform grid,
- construction of the interpolation function,
- construction of circulant integrals,
- determination of the quadrature rule.

Table 4. Iteration counts for example 2 with different preconditioners and the Rectangular rule (* means $>$ 1000 iterations)

	$\tau = 25$			$\tau = 75$			$\tau = 225$		
n	Alg.WR	Alg.SR	Alg.OR	Alg.WR	Alg.SR	Alg.OR	Alg.WR	Alg.SR	Alg.OR
40	117	29	26	251	32	26	763	31	25
80	295	33	31	539	35	30	*	34	31
120	472	40	38	897	41	37	*	43	38
160	*	47	44	*	47	42	*	47	43
200	*	63	57	*	61	57	*	63	58

Table 5. Iteration counts for example 2 with different preconditioners and the Simpson rule (* means $>$ 1000 iterations)

	$\tau = 25$			$\tau = 75$			$\tau = 225$		
n	Alg.WS	Alg.SS	Alg.OS	Alg.WS	Alg.SS	Alg.OS	Alg.WS	Alg.SS	Alg.OS
40	97	26	24	191	27	23	467	27	24
80	261	30	29	459	30	29	*	31	29
120	395	38	36	773	38	37	*	39	37
160	871	42	39	*	40	39	*	41	40
200	*	58	54	*	59	56	*	58	55

Table 6. Number of iterations and the logarithm of condition number for Example 2 with different algorithms

$n \times n$	Alg.	Iteration	$\log_2$(Condition number)	Alg.	Iteration	$\log_2$(Condition number)
10×10		30	1.8×10^1		21	9.1×10^0
20×20	WR	72	2.3×10^1	WS	51	1.3×10^1
30×30		102	8.9×10^1		83	4.9×10^1
10×10		14	2.6×10^{-2}		11	1.3×10^{-2}
20×20	SR	18	7.2×10^{-2}	SS	15	4.5×10^{-2}
30×30		23	9.7×10^{-2}		21	7.8×10^{-2}
10×10		10	0.8×10^{-2}		9	0.6×10^{-2}
20×20	OR	17	5.6×10^{-2}	OS	13	2.6×10^{-2}
30×30		19	8.7×10^{-2}		18	8.3×10^{-2}

The convergence rate of the preconditioned systems and the cost per iteration of the PCG method depend on how we discretize the preconditioning operators. From the numerical results, we see that it is advantageous to use a special nonuniform grid and a quadrature rule to discretize the operator equation of the accuracy concern.

But to speed up the convergence rate of the method and to minimize the cost per iteration, we need to use our a) proposed preconditioner rather than circulant ones and b) proposed interpolation function rather than a special nonuniform grid.

Appendix

The single-layer potential $u = \int_S \frac{\sigma ds}{|x-\xi|}$ has the following properties:

Consider the derivative of u taken in direction of a line normal to the surface S in the outward direction from S. Then, we have

$$\frac{\partial u}{\partial n}|_{p_+} = -2\pi\sigma(p) + \int_S \sigma(Q)\frac{\cos(\xi - x, n)}{|\xi - x|^2}dS, \tag{1}$$

and

$$\frac{\partial u}{\partial n}|_{p_-} = 2\pi\sigma(p) + \int_S \sigma(Q)\frac{\cos(\xi - x, n)}{|\xi - x|^2}dS, \tag{2}$$

where p_+ and p_- signify limits approaching S from G_i and G_e respectively, and where both x and ξ are on S. From (1) and (2), we obtain the jump, the normal derivative, of u across S to be $\sigma = \frac{1}{4\pi}\left(\frac{\partial u}{\partial n}|_{p_-} - \frac{\partial u}{\partial n}|_{p_+}\right)$ (see [8, p. 97]) for instance).

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