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Hypercomplex mathematics and HPM for the time-delayed Burgers equation with convergence analysis

Davood Rostamy · K. Karimi

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Abstract We investigate the analytical and numerical solutions of the time-delayed Burgers equation, by applying the idea of commutative hypercomplex mathematics and the homotopy perturbation method. Moreover, we discuss at great length the convergence conditions of the homotopy perturbation Method (HPM) by using the Banach fixed point theory , which could provide a good iteration algorithm. Finally, we also give some numerical illustrations to the obtained results.

Keywords Time-delayed Burgers equation · Commutative hypercomplex mathematics · Homotopy perturbation method

1 Introduction

The HPM method is an easy technique for finding a power series as a solution of nonlinear differential equations [3, 12–15]. At present this method is very popular and we can find a lot of publications with applications of this method in journals. We can observe a number of ecstatic words about possibilities of this method. It is also shown that the method, with the help of series, provides a powerful mathematical tool for solving other nonlinear evolution equations

D. Rostamy (✉) · K. Karimi
Department of Mathematics, Imam Khomeini International University, Qazvin, Iran
e-mail: rostamy@khayam.ut.ac.ir

K. Karimi
e-mail: Kobra.karimi@yahoo.com

D. Rostamy
Department of Mathematics, Islamic Azad University, South Tehran Branch, Tehran, Iran

arising in mathematical physics. But we observe that HPM does not tell us anything about the possibility of convergence analysis. This paper emphasize the need for convergence conditions.

On the other hand, many phenomena in physics and other fields such as biology, chemistry, mechanics, etc., are described by time delay differential equations [9, 16]. They arise when the rate of change of a time-dependent process in its mathematical modeling is not only determined by its present state but also by a certain past state. Recent studies such as dynamics [10], chemical [6], economical and biological processes [11] have shown that delay differential equation plays an important role in explaining many different phenomena. One of the well know delay differential equations which is an important feature in reaction-diffusion and convection-diffusion systems [1, 7, 8] are time delayed Burgers equation. Moreover, the generalized time-delayed Burgers equation is presented in [8, 16]. It takes the following form:

$$A_s(u) = f(u), \tag{1}$$

where

$$A_s(u) = \tau u_{tt} + [1 - \tau f_u]u_t - u_{xx} + pu^s u_x,$$

and

$$f(u) = qu(1-u),$$

such that $\tau, p, q \in \mathbb{R}$ and $s \in \mathbb{N}$ (see [7]).

In this paper, there's an emphasis on assessing the convergence analysis for HPM on the above problem. Hence, we try to obtain the classical solutions by a new idea that is the commutative hypercomplex mathematics [4, 5]. Therefore, this paper is organized as follows:

In Section 2, we introduce hypercomplex mathematics and in Section 3, we use hypercomplex mathematics to obtain solutions of the time-delayed Burgers equation. In Section 4, we recall idea of homotopy method and in Section 5, we verify the contraction mapping in homotopy method for this problem. The numerical solutions of the problem are obtained by using homotopy perturbation method in Section 6, also, we compare the numerical and exact results in this section. Finally, Section 7 is devoted to the conclusion remarks.

2 Review of commutative hypercomplex mathematics

Systems of hypercomplex numbers, which had been studied and developed at the end of the 19th century, are nowadays quite unknown to the scientific community. It is believed that study of their applications ended just before one of the fundamental discoveries of the 20th century, Einstein's equivalence between space and time. Owing to this equivalence, not-defined quadratic forms have got concrete physical meaning and have been recently recognized to be in strong relationship with a system of bi-dimensional hypercomplex numbers. The commutative hypercomplex mathematics is an extension of complex numbers that obeys the axioms of the classical complex variables. It is 4-D independent variable, so we will use the notation $\mathbf{Z} = 1x + \mathbf{i}y + \mathbf{j}z + \mathbf{k}ct$, where x, y, z, ct are real and $\mathbf{Z}$ belong to an element of the commutative hypercomplex algebra. In the fourth component, t represents time, and c is a scale factor [4, 5].

Analytic function is defined as following:

$$u(\mathbf{Z}) = u(\xi)\mathbf{e}_1 + u(\eta)\mathbf{e}_2,$$
$$\xi = (x - ct) + \mathbf{i}(y + z),$$
$$\eta = (x + ct) + \mathbf{i}(y - z),$$
$$\mathbf{e}_1 = \left(\frac{1 - \mathbf{k}}{2}\right), \quad \mathbf{e}_2 = \left(\frac{1 + \mathbf{k}}{2}\right).$$

The 4-D function $u(\mathbf{Z})$ is analytic if both $u(\xi)$ and $u(\eta)$ are analytic in the classical complex variable sense. Now, we introduce operators such as derivative and integral for functions of a 4-D variable. They obey the function definition that we already have. Therefore, they are as following:

$$\text{oper}(\mathbf{Z}) = \text{oper}(\xi)\mathbf{e}_1 + \text{oper}(\eta)\mathbf{e}_2.$$

The result is that we can apply all of the powerful tools of complex analysis to four-space problems.

The 4-D Cauchy-Riemann(C-R) conditions which have a number of interesting is:

$$\frac{du}{d\mathbf{Z}} = 1\frac{\partial u}{\partial x} = -\mathbf{i}\frac{\partial u}{\partial y} = -\mathbf{j}\frac{\partial u}{\partial z} = \mathbf{k}\frac{\partial u}{\partial ct}. \tag{2}$$

C-R conditions say that the derivative of a 4-D analytic function is the same within a sign in all four coordinate directions. The first two equalities are the same as for complex variables. These equations can be used to reduce a partial differential equation in several real, independent variables to an ordinary differential equation in one 4-D variable. By doing so, we would be imposing continuity conditions on the PDE, because the C-R conditions are a statement of continuity. PDEs are typically derived with the assumption of continuity, but without its explicit inclusion because convenient means have not been available. Note carefully that we are not constraining any potential solution, because the C-R conditions hold for any and all analytic functions.

3 Exact solutions of the time-delayed Burgers equation

It is also shown that the hypercomplex mathematics, with the help of symbolic computation, provides a powerful mathematical tool for solving other nonlinear evolution equations arising in mathematical physics. In this section we impose a proposition that says how to get the analytic solutions of the time-delayed Burgers equation.

Theorem 3.1 *The general analytical solution of* (1) *in 4-D space is as follows:*

$$u(\mathbf{Z}) = \frac{-\sqrt{3}\beta(u(\mathbf{Z}))}{3q(s+1)\left(\frac{1}{2}-\mathbf{k}c\right)} \tan h \left[\frac{\sqrt{3}\beta(u(\mathbf{Z}))(\mathbf{Z}+\mathbf{B})}{3(s+1)(\tau c^2-1)} \right].$$

For $q = 0$ *and* $s = 1$ *we have:*

$$u(\mathbf{Z}) = -\frac{\mathbf{k}c}{p} + \sqrt{\frac{2\mathbf{A}}{p} - \frac{c^2}{p^2}} \tan \left[\frac{\sqrt{\frac{2\mathbf{A}}{p} - \frac{c^2}{p^2}}}{-\frac{2}{p}(\tau c^2 - 1)} (\mathbf{Z}+\mathbf{B}) \right].$$

Where **A**, **B** *are the arbitrary 4-D constants of integration,* **k** *is a 4-D algebraic basis element , c is a scale factor and*

$$\beta(u(\mathbf{Z})) = \sqrt{\left(3pu^{s+1}(\mathbf{Z}) + qu^3(\mathbf{Z})s + qu^3(\mathbf{Z}) - 3\mathbf{k}cu(\mathbf{Z})s - 3\mathbf{k}cu(\mathbf{Z}) - 3\mathbf{A}s - 3\mathbf{A}\right) q \left(\frac{1}{2} - \mathbf{k}c\right)(s+1)}.$$

Proof As mentioned previous section, our basic approach to solution is to first convert the (nonlinear) time-delayed Burgers equation to an ODE, then solve it by means of classical methods. To convert partial differentials to ordinary derivatives, we shall use the 4-D Cauchy–Riemann equations (2), where $\mathbf{Z} = 1x + \mathbf{i}y + \mathbf{j}z + \mathbf{k}ct$. Making the partial derivative conversions, (1) converts to

$$\tau c^2 \frac{d^2u(\mathbf{Z})}{d\mathbf{Z}^2} + \mathbf{k}c\frac{du(\mathbf{Z})}{d\mathbf{Z}}(1-q(1-2u(\mathbf{Z}))) - \frac{d^2u(\mathbf{Z})}{d\mathbf{Z}^2} + pu^s(\mathbf{Z})\frac{du(\mathbf{Z})}{d\mathbf{Z}} = qu(\mathbf{Z})(1-u(\mathbf{Z})).$$

This equation is nonlinear ordinary differential equation, but is solvable by direct methods. A first integration yields:

$$\left(\tau c^2 - 1\right)\frac{du(\mathbf{Z})}{d\mathbf{Z}} + \mathbf{k}cu(\mathbf{Z})(1-q+u(\mathbf{Z})) + \frac{p}{s+1}u^{s+1}(\mathbf{Z}) = q\left(\frac{1}{2}u^2(\mathbf{Z}) - \frac{1}{3}u^3(\mathbf{Z})\right) + \mathbf{A},$$

where **A** is an arbitrary 4-D constant of integration. This is again integrable. We complete the square on the left, move like terms to separate sides of the equation, then integrate to get:

$$\frac{d\mathbf{Z}}{\tau c^2 - 1} = \frac{du(\mathbf{Z})}{\frac{-p}{s+1}u^{s+1}(\mathbf{Z}) - \frac{q}{3}u^3(\mathbf{Z}) + q\left(\frac{1}{2}-\mathbf{k}c\right)u^2(\mathbf{Z}) + \mathbf{k}cu(\mathbf{Z}) + \mathbf{A}}, \quad (3)$$

and we have

$$\frac{\mathbf{Z}+\mathbf{B}}{\tau c^2-1} = \frac{(s+1)\sqrt{3}\operatorname{arctan}h\left(\frac{-\sqrt{3}q\left(\frac{1}{2}-\mathbf{k}c\right)(s+1)u(\mathbf{Z})}{\sqrt{(3pu^{s+1}(\mathbf{Z})+qu^3(\mathbf{Z})s+qu^3(\mathbf{Z})-3\mathbf{k}cu(\mathbf{Z})s-3\mathbf{k}cu(\mathbf{Z})-3\mathbf{A}s-3\mathbf{A})q\left(\frac{1}{2}-\mathbf{k}c\right)(s+1)}}\right)}{\sqrt{\left(3pu^{s+1}(\mathbf{Z})+qu^3(\mathbf{Z})s+qu^3(\mathbf{Z})-3\mathbf{k}cu(\mathbf{Z})s-3\mathbf{k}cu(\mathbf{Z})-3\mathbf{A}s-3\mathbf{A}\right)q\left(\frac{1}{2}-\mathbf{k}c\right)(s+1)}} \tag{4}$$

where $\mathbf{B}$ is another arbitrary 4-D constant of integration. Therefore, we have:

$$u(\mathbf{Z}) = \frac{-\sqrt{3}\beta(u(\mathbf{Z}))}{3q(s+1)\left(\frac{1}{2}-\mathbf{k}c\right)}\tan h\left[\frac{\sqrt{3}\beta(u(\mathbf{Z}))(\mathbf{Z}+\mathbf{B})}{3(s+1)(\tau c^2-1)}\right]. \tag{5}$$

Equation (5) can be converted to the special case when we put $s=1$ and $q=0$:

$$u(\mathbf{Z}) = -\frac{\mathbf{k}c}{p}+\sqrt{\frac{2\mathbf{A}}{p}-\frac{c^2}{p^2}}\tan\left[\frac{\sqrt{\frac{2\mathbf{A}}{p}-\frac{c^2}{p^2}}}{-\frac{2}{p}\left(\tau c^2-1\right)}(\mathbf{Z}+\mathbf{B})\right]. \tag{6}$$

Equation (6) is dependent upon the condition $\frac{2\mathbf{A}}{p}-\frac{c^2}{p^2} > (0,0,0,0)$. □

Above theorem gives us the general analytical solution for the time-delayed Burgers equation in terms of the 4-D commutative hypercomplex variable $\mathbf{Z}$. It is the complete solution for the ODE form in as much as we have integrated twice and have a solution including two arbitrary constants of integration.

We have obtained a solution $u(\mathbf{Z})$ in terms of one variable having three space dimensions and time. One question that must be answered is, "Does it reduce to a solution of the original, one-dimensional time-delayed Burgers equation when the y, z components are set to zero?", this is enough to check. Setting $y=z=0$ in (5) and (6), we get:

$$u(x,ct) = \frac{-\sqrt{3}\beta(u(x,ct))}{3q(s+1)\left(\frac{1}{2}-\mathbf{k}c\right)}\tan h\left[\frac{\sqrt{3}\beta(u(x,ct))(x+\mathbf{k}ct+\mathbf{B})}{3(s+1)(\tau c^2-1)}\right], \tag{7}$$

and

$$u(x,ct) = -\frac{\mathbf{k}c}{p}+\sqrt{\frac{2\mathbf{A}}{p}-\frac{c^2}{p^2}}\tan\left[\frac{\sqrt{\frac{2\mathbf{A}}{p}-\frac{c^2}{p^2}}}{-\frac{2}{p}(\tau c^2-1)}(x+\mathbf{k}ct+\mathbf{B})\right], \tag{8}$$

where

$$\beta(u(x,ct)) = \sqrt{\left(3pu^{s+1}(x,ct)+qu^3(x,ct)s+qu^3(x,ct)-3\mathbf{k}cu(x,ct)s-3\mathbf{k}cu(x,ct)-3\mathbf{A}s-3\mathbf{A}\right)q\left(\frac{1}{2}-\mathbf{k}c\right)(s+1)}.$$

For (8), taking the requisite partial derivatives of $u(x,ct)$ in the usual way, we have:

$$u_t = \mathbf{k}c\frac{\frac{2\mathbf{A}}{p}-\frac{c^2}{p^2}}{\frac{-2}{p}(\tau c^2-1)}\sec^2\left[\frac{\sqrt{\frac{2\mathbf{A}}{p}-\frac{c^2}{p^2}}}{\frac{-2}{p}(\tau c^2-1)}(x+\mathbf{k}ct+\mathbf{B})\right],$$

$$\tau u_{tt} = \mathbf{k}c\frac{\sqrt{\frac{2\mathbf{A}}{p}-\frac{c^2}{p^2}}}{\frac{-2}{p}(\tau c^2-1)}\tan\left[\frac{\sqrt{\frac{2\mathbf{A}}{p}-\frac{c^2}{p^2}}}{-\frac{2}{p}(\tau c^2-1)}(x+\mathbf{k}ct+\mathbf{B})\right]\mathbf{k}c\tau\frac{\frac{2\mathbf{A}}{p}-\frac{c^2}{p^2}}{\frac{-1}{p}(\tau c^2-1)}$$

$$\times\sec^2\left[\frac{\sqrt{\frac{2\mathbf{A}}{p}-\frac{c^2}{p^2}}}{-\frac{2}{p}(\tau c^2-1)}(x+\mathbf{k}ct+\mathbf{B})\right],$$

$$-puu_x = \left(-\frac{\mathbf{k}c}{p}+\sqrt{\frac{2\mathbf{A}}{p}-\frac{c^2}{p^2}}\tan\left[\frac{\sqrt{\frac{2\mathbf{A}}{p}-\frac{c^2}{p^2}}}{\frac{-2}{p}(\tau c^2-1)}(x+\mathbf{k}ct+\mathbf{B})\right]\right)$$

$$\times p\left(\frac{\frac{2\mathbf{A}}{p}-\frac{c^2}{p^2}}{\frac{2}{p}(\tau c^2-1)}\sec^2\left[\frac{\sqrt{\frac{2\mathbf{A}}{p}-\frac{c^2}{p^2}}}{-\frac{2}{p}(\tau c^2-1)}(x+\mathbf{k}ct+\mathbf{B})\right]\right),$$

$$u_{xx} = \frac{\frac{2\mathbf{A}}{p}-\frac{c^2}{p^2}}{\frac{2}{p}(\tau c^2-1)}\tan\left[\frac{\sqrt{\frac{2\mathbf{A}}{p}-\frac{c^2}{p^2}}}{-\frac{2}{p}(\tau c^2-1)}(x+\mathbf{k}ct+\mathbf{B})\right]$$

$$\times\frac{\sqrt{\frac{2\mathbf{A}}{p}-\frac{c^2}{p^2}}}{-\frac{1}{p}(\tau c^2-1)}\sec^2\left[\frac{\sqrt{\frac{2\mathbf{A}}{p}-\frac{c^2}{p^2}}}{-\frac{2}{p}(\tau c^2-1)}(x+\mathbf{k}ct+\mathbf{B})\right].$$

If we combine these equations by addition, then we get the following one-dimensional time-delayed Burgers equation:

$$A(u) = 0, \tag{9}$$

where $A(u) = A_1(u) = \tau u_{tt} + [1-\tau f_u]u_t - u_{xx} + pu^1u_x$. We can obtain (1) if we repeat the above process for (7).

4 The Homotopy Perturbation Method (HPM)

In this section, we introduce the homotopy perturbation method and discuss the convergence analysis. To illustrate the basic ideas of the homotopy perturbation method, we consider the following nonlinear differential equation:

$$T(u) - F(r) = 0, \quad r \in \Omega, \tag{10}$$

with the boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{11}$$

where T is a general differential operator, B is a boundary operator, $F(r)$ is a known analytical function and Γ is the boundary of the domain Ω. Generally speaking, the operator T can be divided into two parts which are L and N, where L is linear, but N is nonlinear. Therefore (10) can be rewritten as follows:

$$L(u) - N(u) - F(r) = 0. \tag{12}$$

By the homotopy perturbation technique, we construct a homotopy $v(r, \theta) : \Omega \times [0, 1] \longrightarrow \mathbb{R}$ with satisfies

$$H(v, \theta) = (1-\theta)[L(v) - L(u_0)] + \theta[T(v) - F(r)] = 0, \quad \theta \in [0, 1], r \in \Omega, \tag{13}$$

or

$$H(v, \theta) = L(v) - L(u_0) + \theta L(u_0) + \theta[N(v) - F(r)] = 0, \tag{14}$$

where $\theta \in [0, 1]$ is an embedding parameter and u_0 is an initial approximation of (10). Obviously, from these definitions we will have

$$H(v, 0) = L(v) - L(u_0) = 0, \quad H(v, 1) = T(v) - F(r) = 0.$$

The process of changing of θ from zero to unity is just that of $v(r, \theta)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(v) - L(u_0)$ and $T(v) - F(r)$ are called homotopies. According to the HPM, we can first use the embedding parameter θ as a "small paramter", and assume that the solution of (13) and (14) can be written as a power series in θ:

$$v = v_0 + \theta v_1 + \theta^2 v_2 + \ldots. \tag{15}$$

Setting $\theta = 1$ results in the approximate solution of (10):

$$u = \lim_{\theta \to 1} v = v_0 + v_1 + v_2 + \ldots.$$

5 Contraction mapping in HPM

The simplicity of contraction mapping in HPM for (1) is the fact that it is uncomplicated and can be understood easily if we can prove it for (9). Therefore, in this section, we apply the homotopy perturbation method for

(9), after that we present an example to show the efficiency and high accuracy of the described method for solving (9).

In order to solve (9) by means of homotopy perturbation method, according to (13), we can construct a convex homotopy such that

$$H(v,\theta)=(1-\theta)\left[\tau\frac{\partial^2 v}{\partial t^2}-\tau\frac{\partial^2 u_0}{\partial t^2}\right]+\theta\left[\tau\frac{\partial^2 v}{\partial t^2}+\frac{\partial v}{\partial t}-\frac{\partial^2 v}{\partial x^2}+pv\frac{\partial v}{\partial x}\right]=0, \quad (16)$$

substituting (15) in (16) and equating the coefficient of like powers of θ yield:

$$\theta^0 : \tau\frac{\partial^2 v_0}{\partial t^2}-\tau\frac{\partial^2 u_0}{\partial t^2}=0,$$

$$\theta^1 : \tau\frac{\partial^2 v_1}{\partial t^2}+\frac{\partial v_0}{\partial t}+pv_0\frac{\partial v_0}{\partial x}-\frac{\partial^2 v_0}{\partial x^2}=0,$$

$$\theta^2 : \tau\frac{\partial^2 v_2}{\partial t^2}+\frac{\partial v_1}{\partial t}+pv_1\frac{\partial v_0}{\partial x}+pv_0\frac{\partial v_1}{\partial x}-\frac{\partial^2 v_1}{\partial x^2}=0,$$

$$\vdots$$

$$\theta^n : \tau\frac{\partial^2 v_n}{\partial t^2}+\frac{\partial v_{n-1}}{\partial t}+p\sum_{i=0}^{n-1}v_i\frac{\partial v_{(n-1)-i}}{\partial x}-\frac{\partial^2 v_{n-1}}{\partial x^2}=0. \quad (17)$$

Then starting with an initial approximation and solving the above equations, we get the nth approximation of the exact solution as $u_n = v_0 + v_1 + v_2 + \cdots + v_n$. On the other hand for investigating of convergence analysis, we rearrangement (17) to the following iterative method:

$$v_j=\frac{-1}{\tau}\oint_{\Omega}\left(\frac{\partial v_{j-1}}{\partial t}+p\sum_{i=0}^{n-1}v_i\frac{\partial v_{(j-1)-i}}{\partial x}-\frac{\partial^2 v_{j-1}}{\partial x^2}\right)d\Omega=A_{j-1}(v_{j-1}), \quad (18)$$

where $j = 1, 2, 3, ..., n$. We show that the above iterative method is a contraction.

The following definition [2] is imposed, then convergence of the homotopy perturbation method is discussed.

Definition 5.1 Let k be a non-negative integer, $r \in [1,\infty)$. Then Sobolev space $W^{k,r}(\Omega)$ is the set of all the functions $v \in L^r(\Omega)$ such that for each multi-index α with $|\alpha| \le k$, the α^{th} weak derivative $\partial^\alpha v$ exists and $\partial^\alpha v \in L^r(\Omega)$. The norm in the space $W^{k,r}(\Omega)$ is defined as

$$\|v\|_{W^{k,r}(\Omega)}=\begin{cases}\left[\sum_{|\alpha|\le k}\|\partial^\alpha v\|^r_{L^r(\Omega)}\right]^{1/r}, & 1\le r<\infty,\\ \max_{|\alpha|\le k}\|\partial^\alpha v\|_{L^\infty(\Omega)}, & p=\infty.\end{cases}$$

Theorem 5.2 *In* (18) *we define, $A_j : W^{k,r} \to W^{k,r}$ and if A'_j, A''_j are bounded in some neighborhood. Then* (i) *A_j is a contraction mapping, that is*

$$\forall v, w \in W^{k,r}; \quad \|A_j(v)-A_j(w)\|_{W^{k,r}} \le \gamma\|v-w\|_{W^{k,r}}, \quad 0<\gamma<1.$$

On the other hand, according to Banach's fixed point theorem, having the fixed point u, that is $u = A_j(u)$. Assume that the sequence generated by homotopy perturbation method can be written as

$$V_n = A_j(V_{n-1}), \quad V_{n-1} = \sum_{i=0}^{n-1} u_i, \quad n = 1, 2, 3, \ldots,$$

and suppose that $V_0 = v_0 = u_0$,then,

(ii) *The sequence of $\{V_n\}_{n=1}^{\infty}$ is convergence, i.e.,*

$$\exists u \in W^{k,r}, \quad \lim_{n\to\infty} V_n = u.$$

(iii) *u in* (ii) *is the exact solution.*

Proof

(i) By using Definition 5.3, Lemmas 5.4 and 5.5 and Theorem 5.6 we can conclude this claim.

(ii) It is enough to show that, $\{V_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the Sobolev space. For this reason, consider, for every $n, m \in \mathbb{N}$, $n \geq m$, we have

$$\begin{aligned}
\|V_n - V_m\|_{W^{k,r}} &= \|(V_n - V_{n-1}) + (V_{n-1} - V_{n-2}) \\
&\quad + \cdots + (V_{m+1} - V_m)\|_{W^{k,r}} \\
&\leq \|V_n - V_{n-1}\|_{W^{k,r}} + \|V_{n-1} - V_{n-2}\|_{W^{k,r}} \\
&\quad + \cdots + \|V_{m+1} - V_m\|_{W^{k,r}} \\
&= \|A_j(V_{n-1}) - A_j(V_{n-2})\|_{W^{k,r}} + \|A_j(V_{n-2}) \\
&\quad - A_j(V_{n-3})\|_{W^{k,r}} + \cdots + \|A_j(V_m) - A_j(V_{m-1})\|_{W^{k,r}} \\
&\leq \gamma^{n-1}\|V_1 - v_0\|_{W^{k,r}} + \gamma^{n-2}\|V_1 - v_0\|_{W^{k,r}} \\
&\quad + \cdots + \gamma^{m}\|V_1 - v_0\|_{W^{k,r}} \\
&\leq (\gamma^m + \gamma^{m+1} + \ldots)\|V_1 - v_0\|_{W^{k,r}} = \frac{\gamma^m}{1-\gamma}\|V_1 - v_0\|_{W^{k,r}},
\end{aligned}$$

Since $\|V_1 - v_0\|_{W^{k,r}} < \infty$, hence, $\lim_{n.m\to+\infty} \|V_n - V_m\|_{W^{k,r}} = 0$, i.e., $\{V_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the Sobolev space $W^{k,r}$ and it implies that

$$\exists u \in W^{k,r}, \quad \lim_{n\to+\infty} V_n = u.$$

(iii) Using (ii), we have

$$A_j(u) = A_j\left(\lim_{n\to+\infty} V_n\right) = \lim_{n\to+\infty} A_j(V_n) = \lim_{n\to+\infty} V_{n+1} = u,$$

i.e., u is a solution of (9).

□

We can split the operator of A_j to the following linear and nonlinear operators:

$$A_j(u) = A_{L,j}(u) + A_{N,j}(u),$$

where $A_{L,j}$, $A_{N,j}$ are linear and nonlinear operators respectively. Moreover, We use the idea of the derivative of an operator by generalizing the definition for the derivative of a function of one variable. Hence, this generalization can be made in different ways. We are led to several possible definitions for the derivative of an operator but in this paper we will use the definition of *Frechet derivative*. In the following we recall it:

Definition 5.3 Let $A_j : W^{k,r} \to W^{k,r}$ and $\overline{u} \in W^{k,r}$ and there exists a bounded linear operator $A'_j(\overline{u})$ such that for all $\|\Delta u\|_{W^{k,r}} \longrightarrow 0$

$$\lim_{\|\Delta u\|_{W^{k,r}} \longrightarrow 0} \frac{\|A_j(\overline{u} + \Delta u) - A_j(\overline{u}) - A'_j(\overline{u})\Delta u\|_{W^{k,r}}}{\|\Delta u\|_{W^{k,r}}} = 0$$

then we say that A_j is strongly differentiable at $\overline{u}$. The operator $A'_j(\overline{u})$ is called the strong or *Frechet derivative*.

We can also define the higher order of differentiate it (see [2]). In elementary numerical analysis Taylor's theorem is frequently used in analyzing algorithms. For the operator of A_j there exists an extension of Taylor's Theorem which is equally used. In the following we highlight some lemmas about it.

Lemma 5.4 *The second Frechet derivative of $A_{L,j}$ is zero that is for every $\overline{u} \in W^{k,r}$ we have $A'_{L,j}(\overline{u}) = A_{L,j}$.*

Proof It is obvious from the above definition. Of course, we note that this does not say that $A'_{L,j}$ and $A_{L,j}$ are identical, but that $A'_{L,j}$ has the same value A_L at all points $\overline{u} \in W^{k,r}$. The observation is analogous to the fact that the derivative of linear operator is a constant and we have $A''_{L,j} = 0$. □

Lemma 5.5 *The third Frechet derivative of $A_{N,j}$ is zero.*

Proof Consider the operator of $A_{N,0}(u) = p \oint_\Omega u u_x d\Omega$, we can write the following operator from the above definition:

$$A'_{N,0}(\overline{u})(\quad) = p \oint_\Omega ((\overline{u_x})(\quad) + (\overline{u})(\quad)_x) d\Omega,$$

$$A''_{N,0}(\overline{u})(\quad)(\quad) = 2p \oint_\Omega (\quad)_x(\quad) d\Omega,$$

$$A'''_{N,0}(\overline{u}) = 0,$$

as is easily verified. We can repeat the above proof for every $A_{N,j}$, $j = 1, 2, ..., n-1$ by induction. □

Theorem 5.4 *The operator of A_j in (18) is a contraction mapping if $\gamma < 1$, A'_j and A''_j are bounded in some neighborhood.*

Proof we can write:

$$\begin{aligned}\|A_j(u) - A_j(\overline{u})\|_{W^{k,r}} &= \|A_j(u) - A_j(\overline{u}) - A'_j(u)(u-\overline{u}) + A'_j(u)(u-\overline{u})\|_{W^{k,r}} \\ &\leq \|A_j(u) - A_j(\overline{u}) - A'_j(u)(u-\overline{u})\|_{W^{k,r}} \\ &\quad + \|A'_j(u)(u-\overline{u})\|_{W^{k,r}}.\end{aligned}$$

According to the theorem of generalized Taylor (see [11]) and the above lemmas, we can write the following inequality, where $l_j(\overline{u}, u)$ is the line segment between $\overline{u}, u$:

$$\begin{aligned}\|A_j(u) - A_j(\overline{u})\|_{W^{k,r}} &\leq \sup_{u\in l_j(\overline{u},u)} \|A''_{N,j}(\overline{u})\|_{W^{k,r}} \frac{\|u-\overline{u}\|^2_{W^{k,r}}}{2} \\ &\quad + \|A'_j(\overline{u})\|_{W^{k,r}} \|u-\overline{u}\|_{W^{k,r}} \leq \gamma \|u-\overline{u}\|_{W^{k,r}},\end{aligned}$$

where $\gamma = \sup_{u\in l_j(\overline{u},u)} \|A''_{N,j}(\overline{u})\|_{W^{k,r}} \frac{\|u-\overline{u}\|_{W^{k,r}}}{2} + \|A'_j(\overline{u})\|_{W^{k,r}}$. □

6 Numerical experimental

Example 1 We consider (9) with $\tau = 0.5$, $p = 0.1$ and the initial condition $u(x, 0) = 1 - \tanh(0.05025x)$. According to (18) we have:

$$v_0(x,t) = 1 - \tanh(0.05025x),$$

$$\begin{aligned}v_1(x,t) &= \frac{1}{2}\left(\frac{201}{20000} - \frac{201}{20000}\tanh\left(\frac{201}{4000}x\right)^2 + \frac{201}{4000000}\tanh\left(\frac{201}{4000}x\right)\right. \\ &\quad \left. - \frac{201}{4000000}\tanh\left(\frac{201}{4000}x\right)^3\right)t^2,\end{aligned}$$

$$\begin{aligned}v_2(x,t) &= -\frac{1}{500000000000000}\frac{1}{\cosh\left(\frac{201}{4000}x\right)^5}\left(\frac{1}{12}\left(14\sinh\left(\frac{201}{4000}x\right)\cosh\left(\frac{201}{4000}x\right)^4\right.\right. \\ &\quad - 50502512533\sinh\left(\frac{201}{4000}x\right)\cosh\left(\frac{201}{4000}x\right)^2 + 50\cosh\left(\frac{201}{4000}x\right)^5 \\ &\quad + 256293844\sinh\left(\frac{201}{4000}x\right) + 757518800\cosh\left(\frac{201}{4000}x\right) \\ &\quad \left. - 505012600\cosh\left(\frac{201}{4000}x\right)^3\right)t^4 + \frac{1}{6}\left(10050000000000\cosh\left(\frac{201}{4000}x\right)^3\right.\end{aligned}$$

Table 1 For Example 1, some values of exact and numerical solutions for $t = 0.1$, $s = 1$ and $p = 0.1$

x_i	−60	−40	−20	0	20	40	60	80	100
$u_E(x, t)$	1.99523	1.9649	1.76474	1.00251	0.237353	0.0354437	0.482284e-2	0.647525e-3	0.86781e-4
u_{HPM}	1.99520	1.9647	1.76376	1.00014	0.236359	0.0352796	0.480038e-1	0.644537e-3	0.86385e-4

$$
\begin{aligned}
&+ 50250000000 \sinh\left(\frac{201}{4000}x\right)\cosh\left(\frac{201}{4000}x\right)^2 t^3\Bigg) \\
&+ \frac{1}{2}\Bigg(-\frac{39999}{4000000}\tanh\left(\frac{201}{4000}x\right) + \frac{39999}{4000000}\tanh\left(\frac{201}{4000}x\right)^3 \\
&\qquad - \frac{201}{10000}\tanh\left(\frac{201}{4000}x\right)^2 + \frac{201}{10000}\Bigg) t^2 \\
&\vdots
\end{aligned}
$$

and so on. In this manner the other components can be easily obtained. In Table 1 we represent the values of the exact $u_E(x, t)$ and numerical solution u_{HPM} for $t = 0.1$. In the latter two figures we use the notation $|u_E(x, t) - u_{HPM}|$ as error. It is clear from this table and following Fig. 1 that the difference between the exact and the numerical solutions is very small.

We repeat one's deed for $p = 10^{-i}$, $i = 2, 3, ..., 6$ and we get Fig. 2. In these experimental results we observe that near $p = 0$, in the method of HPM we have not stability same as the classical solution.

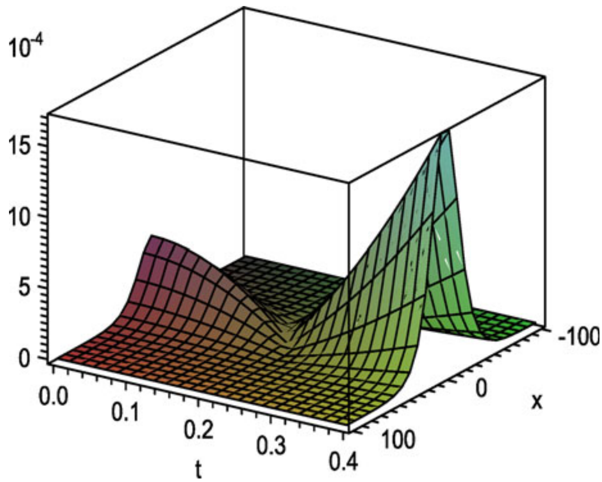

Fig. 1 Error function for the intervals $-100 \leq x \leq 100$, $0 \leq t \leq 0.4$ and $p = 0.1$

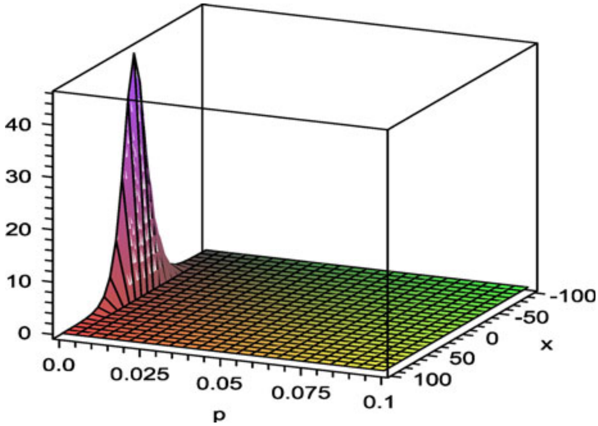

Fig. 2 Error function for the intervals $-100 \le x \le 100$ and $10^{-6} \le p \le 0.1$

Example 2 We consider (1) with $\tau = 0.0$, $p = 0.1$, $s = 2$ and the exact is as follows:

$$u(x,t) = \frac{-3}{6400000000}\frac{1}{\sqrt{6-6\tanh\left(\left(\frac{1}{10}\right)x\right)}} \times \left(-1+\tanh\left(\left(\frac{1}{10}\right)x\right)\right)^2\left(27\tanh\left(\left(\frac{1}{10}\right)x\right)^7+558\tanh\left(\left(\frac{1}{10}\right)x\right)^6 + 21261\tanh\left(\left(\frac{1}{10}\right)x\right)^5+90138\tanh\left(\left(\frac{1}{10}\right)x\right)^4 - 61323\tanh\left(\left(\frac{1}{10}\right)x\right)^3-79110\tanh\left(\left(\frac{1}{10}\right)x\right)^2 + 26835\tanh\left(\left(\frac{1}{10}\right)x\right)+7214\right)\left(\tanh\left(\left(\frac{1}{10}\right)x\right)+1\right)t^3\Big) + \frac{9}{3200000}\frac{1}{\sqrt{6-6\tanh\left(\left(\frac{1}{10}\right)x\right)}} \times \left(\left(\left(-1+\tanh\left(\frac{1}{10}x\right)\right)^2\left(21\left(\tanh\left(\frac{1}{10}x\right)\right)^3+147\left(\tanh\left(\frac{1}{10}x\right)\right)^2 - 81\tanh\left(\frac{1}{10}x\right)-47\right)\left(\tanh\left(\frac{1}{10}x\right)+1\right)t^2\right) - \frac{3}{4000}\frac{t\left(3\tanh\left(\left(\frac{1}{10}\right)x\right)^4+30\tanh\left(\left(\frac{1}{10}\right)x\right)^3-16\tanh\left(\left(\frac{1}{10}\right)x\right)^2-30\tanh\left(\left(\frac{1}{10}\right)x\right)+13\right)}{\sqrt{6-6\tanh\left(\left(\frac{1}{10}\right)x\right)}}$$

$$
\begin{aligned}
&+\sqrt{1.5-1.5\tanh(0.1x)}\\
&+\left(\frac{1}{720000000000000000}\frac{1}{\sqrt{6-6\tanh\left(\left(\frac{1}{10}\right)x\right)}\left(-1+\tanh\left(\left(\frac{1}{10}\right)x\right)\right)}\right.\\
&\quad\times\left(\frac{1}{4}\left(-6031715625+38156315624\tanh\left(\left(\frac{1}{10}\right)x\right)^{2}\right.\right.\\
&\quad+23475318750\tanh\left(\left(\frac{1}{10}\right)x\right)-590417859380\tanh\left(\left(\frac{1}{10}\right)x\right)^{6}\\
&\quad+693973828180\tanh\left(\left(\frac{1}{10}\right)x\right)^{8}-457963875000\tanh\left(\left(\frac{1}{10}\right)x\right)^{7}\\
&\quad+84222281200\tanh\left(\left(\frac{1}{10}\right)x\right)^{9}\\
&\quad-249290662500\tanh\left(\left(\frac{1}{10}\right)x\right)^{3}+131008134380\tanh\left(\left(\frac{1}{10}\right)x\right)^{4}\\
&\quad+571741031250\tanh\left(\left(\frac{1}{10}\right)x\right)^{5}\\
&\quad-295174378120\tanh\left(\left(\frac{1}{10}\right)x\right)^{10}+28342153124\tanh\left(\left(\frac{1}{10}\right)x\right)^{12}\\
&\quad+24084337500\tanh\left(\left(\frac{1}{10}\right)x\right)^{11}\\
&\quad\left.\left.+143521875\tanh\left(\left(\frac{1}{10}\right)x\right)^{14}+3731568750\tanh\left(\left(\frac{1}{10}\right)x\right)^{13}\right)\right)t^{4}\\
&\quad+\left(1/3\left(-27337500000\tanh\left(\left(\frac{1}{10}\right)x\right)^{11}\right.\right.\\
&\quad+262480500000000\tanh\left(\left(\frac{1}{10}\right)x\right)^{4}\\
&\quad+112716562500000\tanh\left(\left(\frac{1}{10}\right)x\right)^{3}\\
&\quad-134439750000000\tanh\left(\left(\frac{1}{10}\right)x\right)^{2}
\end{aligned}
$$

$$- 48265875000000 \tanh\left(\left(\frac{1}{10}\right)x\right)^8$$
$$+ 243516375000000 \tanh\left(\left(\frac{1}{10}\right)x\right)^7$$
$$- 86568750000000 \tanh\left(\left(\frac{1}{10}\right)x\right)^6$$
$$- 348370875000000 \tanh\left(\left(\frac{1}{10}\right)x\right)^5$$
$$+ 12562087500000 \tanh\left(\left(\frac{1}{10}\right)x\right)$$
$$- 20396812500000 \tanh\left(\left(\frac{1}{10}\right)x\right)^9$$
$$- 510300000000 \tanh\left(\left(\frac{1}{10}\right)x\right)^1 0 + 7304175000000\Bigg)\Bigg) t^3\Bigg)\Bigg).$$

According to HPM we have:

$$v_0(x,t) = \sqrt{1.5 - 1.5\tanh(.1x)},$$

$$v_1(x,t) = \frac{-3}{4000}\frac{t\left(3\tanh\left(\left(\frac{1}{10}\right)x\right)^4 + 30\tanh\left(\left(\frac{1}{10}\right)x\right)^3 - 16\tanh\left(\left(\frac{1}{10}\right)x\right)^2 - 30\tanh\left(\left(\frac{1}{10}\right)x\right) + 13\right)}{\sqrt{6 - 6\tanh\left(\left(\frac{1}{10}\right)x\right)}},$$

$$v_2(x,t) = \frac{9}{3200000}\frac{1}{\sqrt{6 - 6\tanh\left(\left(\frac{1}{10}\right)x\right)}}\left(\left(-1 + \tanh\left(\left(\frac{1}{10}\right)x\right)\right)^2\right.$$
$$\times\left(21\tanh\left(\left(\frac{1}{10}x\right)\right)^3 + 147\tanh\left(\left(\frac{1}{10}\right)x\right)^2 - 81\tanh\left(\left(\frac{1}{10}\right)x\right) - 47\right)$$
$$\left.\times\left(\tanh\left(\left(\frac{1}{10}\right)x\right) + 1\right)t^2\right)$$

and so on. In this manner the other components can be easily obtained. In Table 2 we represent the values of the exact $u_E(x,t)$ and numerical solution u_{HPM} for $t = 1.0$.

Table 2 For Example 2, some values of exact and numerical solutions for $t = 1.0$, $s = 2$ and $p = 0.1$

x_i	−40	−30	−20	20	30	40	50
$u_E(x, t)$	1.73177	1.72995	1.71671	.234582	0.869905e-1	0.32037e-1	0.0117875
u_{HPM}	1.731749077	1.729825205	1.715815611	0.2344242928	0.08697856065	0.03203392997	0.01178215219

7 Conclusion remarks

We have found the most general characteristic function for the time-delayed Burgers equation. The equation was solved numerically by the homotopy perturbation method. In our opinion, in some sense that the research results will be of theoretical significance and practical value for constructing the exact as well as approximate solution of nonlinear evolution equations. We observe, that HPM does not tell us anything about the possibility of convergence analysis for HPM but in this paper, we need to suitable conditions. Therefore, one of the shortcoming in this method is removed.

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