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Worst case error for integro-differential equations by a lattice-Nyström method

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Abstract. In this paper, we make an offer of the lattice approximate method for solving a class of multi-dimensional integro-differential equations with the initial conditions. Also, we analyze the worst case error measured in weighted Korobov spaces for these equations. Finally, numerical examples complete this work.

Keywords. QMC-Nyström, lattice quadrature, worst case error, multi-dimensional integral equation, multi-dimensional integro-differential equations.

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1 Introduction

The paper aims to extend the integral equation to the following system of Fredholm integro-differential equations given by

$$p(u) = g(\mathbf{x}) + \lambda_1 \int_D \kappa_1(\mathbf{x}, \mathbf{y}) u(\mathbf{y}) d\mathbf{y} + \lambda_2 \int_D \overrightarrow{\kappa_2}(\mathbf{x}, \mathbf{y}) \cdot \nabla u(\mathbf{y}) d\mathbf{y}, \tag{1.1}$$

with the initial conditions

$$\begin{cases} u(\mathbf{x}) = b_0 & \text{in } \mathbf{x} \in \partial D_1, \\ \text{or} & \\ \nabla u(\mathbf{x}) = \overrightarrow{b_2} & \text{in } \mathbf{x} \in \partial D_2, \end{cases} \tag{1.2}$$

where $\partial D_1, \partial D_2 \subseteq \partial D$ and

$$p(u) = \alpha_0(\mathbf{x}) u(\mathbf{x}) + \overrightarrow{\alpha_1}(\mathbf{x}) \cdot \nabla u(\mathbf{x}) \tag{1.3}$$

for $\mathbf{x} \in D = [0,1]^d$, $\lambda_1, \lambda_2 \in \mathbb{R}$ and $d \in \mathbb{N}$. Also, $\kappa_1, \overrightarrow{\kappa_2}, g, \alpha_0$ and $\overrightarrow{\alpha_1}$ are given functions from a weighted Korobov space. This problem describes several interesting physical and financial phenomena (see [7–9, 13, 14, 22]).

On the other hand, the existence and uniqueness of the above problem were proved by many authors (see [3, 11, 12, 17, 18, 24]). The paper proposes a convergence analysis of discretization schemes for these equations. The convergence analysis is cast into the general setting of information based complexity (IBC). Numerical solution by collocation discretization is proposed, with collocation points chosen as lattice points of a QMC integration rule.

1.1 Preliminary

(See [4, 5, 10].) The weighted Korobov spaces are characterized by a smoothness parameter $\alpha > 1$ and weights $1 \geq \gamma_1 \geq \gamma_2 \geq \cdots > 0$ where γ_j moderates the behavior of the function with respect to the jth variable; a small γ_j means that the function depends weakly on the jth variable. Let $\mathbb{H} = \mathbb{H}^d_{\boldsymbol{\gamma},\alpha}(D)$ denote a weighted Korobov space, where $\boldsymbol{\gamma} = (\gamma_j)_{j \geq 1}$ is a sequence of positive weights and $\alpha > 1$ is a smoothness parameter. For any

$$u(\mathbf{x}) = \sum_{\mathbf{h} \in \mathbb{Z}^d} \hat{u}(\mathbf{h}) e^{2\pi i \mathbf{h}.\mathbf{x}} \quad \text{with} \quad \hat{u}(\mathbf{h}) = \int_D u(\mathbf{x}) e^{-2\pi i \mathbf{h}.\mathbf{x}} d\mathbf{x},$$

the norm of u in $\mathbb{H}$ is given by

$$\|u\|_{\mathbb{H}} = \Big(\sum_{\mathbf{h} \in \mathbb{Z}^d} |\hat{u}(\mathbf{h})|^2 r_a(\boldsymbol{\gamma}, \mathbf{h}) \Big)^{1/2},$$

therefore, if we consider

$$\frac{\partial u}{\partial x_j} = \sum_{\mathbf{h} \in \mathbb{Z}^d} \hat{u}(\mathbf{h}) 2\pi i h_j e^{2\pi i \mathbf{h}.\mathbf{x}} \quad \text{for } j = 1, \ldots, d, \tag{1.4}$$

then we have

$$\|\nabla u(\mathbf{x})\|_{\mathbb{H},\infty} = \Big(\sum_{\mathbf{h} \in \mathbb{Z}^d} \|\hat{u}(\mathbf{h}) 2\pi i \mathbf{h}\|^2_\infty r_\alpha(\boldsymbol{\gamma}, \mathbf{h}) \Big)^{1/2},$$

where

$$r_\alpha(\boldsymbol{\gamma}, \mathbf{h}) = \prod_{j=1}^{d} r_\alpha(\gamma_j, h_j), \tag{1.5}$$

and

$$r_\alpha(\gamma_j, h_j) = \begin{cases} 1 & \text{if } h_j = 0, \\ \gamma_j^{-1} |h_j|^\alpha & \text{otherwise.} \end{cases}$$

Thus, we have $r_\alpha(\boldsymbol{\gamma}, \mathbf{h}) \geq 1$ for all $\mathbf{h} \in \mathbb{Z}^d$. If we use the Cauchy–Schwarz inequality, it shows that for all $u \in \mathbb{H}$ the following inequality is made:

$$\|u\|_{\sup} \leq \sum_{\mathbf{h} \in \mathbb{Z}^d} |\hat{u}(\mathbf{h})| \leq \Big(\sum_{\mathbf{h} \in \mathbb{Z}^d} |\hat{u}(\mathbf{h})|^2 r_\alpha(\boldsymbol{\gamma}, \mathbf{h}) \Big)^{1/2} \Big(\sum_{\mathbf{h} \in \mathbb{Z}^d} \frac{1}{r_\alpha(\boldsymbol{\gamma}, \mathbf{h})} \Big)^{1/2}.$$

Therefore we have

$$\|u\|_{\sup} \le \|u\|_{\mathbb{H}} \prod_{j=1}^{d} \left(1 + 2\zeta(\alpha)\gamma_j\right)^{1/2}, \tag{1.6}$$

where

$$\zeta(x) := \sum_{j=1}^{\infty} j^{-x}$$

denotes the Riemann zeta function. Furthermore the inequality in (1.6) becomes equality when u is a multiple of the function

$$\sum_{\mathbf{h}\in\mathbb{Z}^d} \frac{e^{2\pi i \mathbf{h}.\mathbf{x}}}{r_\alpha(\boldsymbol{\gamma}, \mathbf{h})}.$$

The rest of this paper is organized as follows. The paper investigates two cases of (1.1) in Sections 2 and 3. In these sections, we will obtain that the worst case error achieves the optimal rate of convergence $\mathcal{O}(n^{-\frac{\alpha}{2}+\delta})$, $\delta > 0$, in weighted Korobov spaces, for a sufficiently large n. We assume that $\mathbf{t}_1, \dots, \mathbf{t}_n \in D$ form a set of rank-1 lattice points. On the other hand, tractability in the absolute sense means that the minimal value of n is needed in the Quasi Monte Carlo Nyström (QMC-Nyström) method to reduce the worst case error to $\varepsilon \in (0, 1)$ and it is a bounded polynomial in d and ε^{-1}. Also, the tractability and strong tractability of the QMC-Nyström method in the absolute or normalized sense are investigated. Of course, we know that strong tractability means that the bound is independent of d. Also, we will show that strong QMC-Nyström tractability in the absolute sense holds iff

$$\sum_{j=1}^{\infty} \gamma_j < \infty, \tag{1.7}$$

and QMC-Nyström tractability in the absolute sense holds iff

$$\limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_j}{\log(d+1)} < \infty. \tag{1.8}$$

Moreover, strong tractability in the normalized sense is defined in term of the normalized error with respect to the initial error. The conditions (1.7) and (1.8) are also sufficient conditions for strong QMC-Nyström tractability in the normalized sense [23,25]. Finally, some numerical results for two cases show that the proposed method has merited. Also, some important propositions and results are proved in appendices.

2 First case study of (1.1)

In this section, we will consider $\kappa_1(\mathbf{x},\mathbf{y}) = \kappa(\mathbf{x},\mathbf{y})a_1(\mathbf{y})$, $\overrightarrow{\kappa_2}(\mathbf{x},\mathbf{y}) = \kappa(\mathbf{x},\mathbf{y})\overrightarrow{a_2}(\mathbf{y})$, $\lambda = \lambda_1 = \lambda_2$, $\alpha_0(\mathbf{x}) = 1$ and $\overrightarrow{\alpha_1}(\mathbf{x}) = \overrightarrow{0}$ in (1.1). Therefore we have the following equation:

$$u(\mathbf{x}) = g(\mathbf{x}) + \lambda\left[\int_D \kappa(\mathbf{x},\mathbf{y})a_1(\mathbf{y})u(\mathbf{y})d\mathbf{y} + \int_D \kappa(\mathbf{x},\mathbf{y})\overrightarrow{a_2}(\mathbf{y})\cdot\nabla u(\mathbf{y})d\mathbf{y}\right], \quad (2.1)$$

with the initial conditions (1.2), where the kernel κ is assumed to be of the form $\kappa(\mathbf{x},\mathbf{y}) := k(\mathbf{x}-\mathbf{y})$, with $k(\mathbf{x})$ having period one in each component of $\mathbf{x}$. Further, we assume that g, k belong to a weighted Korobov space $\mathbb{H}$.

We approximate u in (2.1) by using the Nyström method based on QMC rules, that is, equal-weight integration rules. We assume that $\mathbf{t}_1,\ldots,\mathbf{t}_n \in D$ and the approximation of u is given by

$$u_n(\mathbf{x}) := g(\mathbf{x}) + \frac{\lambda}{n}\sum_{i=1}^{n}\big(k(\mathbf{x}-\mathbf{t}_i)a_1(\mathbf{t}_i)u_n(\mathbf{t}_i) + k(\mathbf{x}-\mathbf{t}_i)\overrightarrow{a_2}(\mathbf{t}_i)\cdot\nabla u_n(t_i)\big), \quad (2.2)$$

by differentiating from (2.2), we have

$$\nabla_{\mathbf{x}}u_n(\mathbf{x}) := \nabla_{\mathbf{x}}g(\mathbf{x}) + \frac{\lambda}{n}\sum_{i=1}^{n}\nabla_{\mathbf{x}}k(\mathbf{x}-\mathbf{t}_i)\big(a_1(\mathbf{t}_i)u_n(\mathbf{t}_i) + \overrightarrow{a_2}.(\mathbf{t}_i)\nabla u_n(\mathbf{t}_i)\big), \quad (2.3)$$

where the function values $u_n(\mathbf{t}_1),\ldots,u_n(\mathbf{t}_n)$ are obtained by solving the following linear system from (2.2) and (2.3):

$$\begin{cases} u_n(\mathbf{t}_j) := g(\mathbf{t}_j) + \dfrac{\lambda}{n}\displaystyle\sum_{i=1}^{n}\big(k(\mathbf{t}_j-\mathbf{t}_i)a_1(\mathbf{t}_i)u_n(\mathbf{t}_i) \\ \qquad\qquad + k(\mathbf{t}_j-\mathbf{t}_i)\overrightarrow{a_2}(\mathbf{t}_i)\cdot\nabla u_n(\mathbf{t}_i)\big), \\ \nabla_{\mathbf{x}}u_n(\mathbf{t}_j) := \nabla_{\mathbf{x}}g(\mathbf{t}_j) + \dfrac{\lambda}{n}\displaystyle\sum_{i=1}^{n}\nabla_{\mathbf{x}}k(\mathbf{t}_j-\mathbf{t}_i)\big(a_1(\mathbf{t}_i)u_n(\mathbf{t}_i) \\ \qquad\qquad + \overrightarrow{a_2}(\mathbf{t}_i)\cdot\nabla u_n(\mathbf{t}_i)\big), \end{cases} \quad (2.4)$$

for $j = 1,2,\ldots,n$.

We recall that the system of (2.4) is as a reduced linear system by required conditions given in equations (1.2).

Therefore, in this section we investigate the worst case error of the QMC-Nyström method which essentially the worst possible error $u - u_n$, measured in sup norm. Also, we assume that

$$k_1(\mathbf{x}-\mathbf{y}) := k(\mathbf{x}-\mathbf{y})a_1(\mathbf{y}), \quad \overrightarrow{k_2}(\mathbf{x}-\mathbf{y}) := k(\mathbf{x}-\mathbf{y})\overrightarrow{a_2}(\mathbf{y}).$$

Hence we make a good set of points $\mathbf{t}_1, \ldots, \mathbf{t}_n$ which leads to a as small worst case error as possible. There are alternative methods for tending to this goal, so we select a rank-1 lattice rule as a QMC-rule with points given by $\mathbf{t}_i = \{\frac{i\mathbf{z}}{n}\}$ with $i = 1, 2, \ldots, n$ (see [6,16]). In particular, we need a class of lattice-Nyström methods such that $\mathbf{z}$ is introduced by the generating vector which is an integer vector having no factor in common with n, and the braces around a vector indicate that each component of the vector is to be replaced by its fractional part.

2.1 Lower and upper bounds on the worst case error for (2.1)

Let $\mathcal{C}^{1,v}(D)$ be a set of continuous functions $u : D \to \mathbb{R}$ which are one time continuously differentiable in D and such that for all $t \in D\backslash\partial D$, the following estimate holds:

$$\|\nabla u\| \leq c(u) \begin{cases} 1 & \text{if } v < 0, \\ 1 + |\log \rho(t)| & \text{if } v = 0, \\ \rho^{-v}(t) & \text{if } v > 0, \end{cases}$$

where $-\infty < v < 1$, $c(u)$ is a positive constant and $\rho(t) = \min_{0<t<1}\{t, 1-t\}$ is the distance from $t \in (0,1)$ to the boundary of the interval $(0,1)$. We equip the space of bounded linear operators from $\mathcal{C}^{1,v}(D)$ to $\mathcal{C}^{1,v}(D)$ with the usual operator norm

$$\|K\| = \sup_{\|u\|_{\sup} \leq 1} \|Ku\|_{\sup},$$

for a given kernel, we are interested in the integral operator

$$K : \mathcal{C}^{1,v}(D) \to \mathcal{C}^{1,v}(D)$$

given by

$$\begin{aligned} Ku &= \int_D \kappa(\mathbf{x}, \mathbf{y})\big(a_1(\mathbf{y})u(\mathbf{y}) + \overrightarrow{a_2}(\mathbf{y}).\nabla u(\mathbf{y})\big)d\mathbf{y} \\ &= \int_D \kappa_1(\mathbf{x}, \mathbf{y})u(\mathbf{y})d\mathbf{y} + \int_D \kappa_3(\mathbf{x}, \mathbf{y})d\mathbf{y}, \end{aligned}$$

where $\kappa_3(\mathbf{x}, \mathbf{y}) = \overrightarrow{\kappa_2}(\mathbf{x}, \mathbf{y}) \cdot \nabla u(\mathbf{y})$ and $\overrightarrow{\kappa_2}(\mathbf{x}, \mathbf{y}) = \kappa_2(\mathbf{x}, \mathbf{y})\overrightarrow{a_2}(\mathbf{y})$ with

$$\|K\| = \max_{\mathbf{x} \in D} \int_D |\kappa_1(\mathbf{x}, \mathbf{y})|d\mathbf{y} + \max_{\mathbf{x} \in D} \int_D |\kappa_3(\mathbf{x}, \mathbf{y})|d\mathbf{y},$$

and the corresponding discrete operator

$$K_n : \mathcal{C}^{1,v}(D) \to \mathcal{P}_n$$

given by

$$K_n u = \frac{1}{n}\sum_{i=1}^{n} k(\mathbf{x}-\mathbf{t}_i)\big(a_1(\mathbf{t}_i)u(\mathbf{t}_i) + \overrightarrow{a_2}(\mathbf{t}_i)\cdot \nabla u(\mathbf{t}_i)\big)$$
$$= \frac{1}{n}\sum_{i=1}^{n} k_1(\mathbf{x}-\mathbf{t}_i)u(\mathbf{t}_i) + \frac{1}{n}\sum_{i=1}^{n} \overrightarrow{k_2}(\mathbf{x}-\mathbf{t}_i)\cdot \nabla u(\mathbf{t}_i),$$

with

$$\|K_n\| = \max_{\mathbf{x}\in D}\frac{1}{n}\sum_{i=1}^{n}|k_1(\mathbf{x}-\mathbf{t}_i)| + \max_{x\in D}\frac{1}{n}\sum_{i=1}^{n}|k_3(\mathbf{x}-\mathbf{t}_i)|,$$

where $\mathbf{t}_1,\dots,\mathbf{t}_n \in D$. Thus

$$\|K\| = \int_D |k_1(\mathbf{y})|d\mathbf{y} + \int_D |k_3(\mathbf{y})|d\mathbf{y} \le \|k_1\|_{\sup} + \|k_3\|_{\sup}$$

and

$$\|K_n\| = \max_{x\in D}\frac{1}{n}\sum_{i=1}^{n}|k_1(\mathbf{x}-\mathbf{t}_i)| + \max_{\mathbf{x}\in D}\frac{1}{n}\sum_{i=1}^{n}|k_3(\mathbf{x}-\mathbf{t}_i)| \le \|k_1\|_{\sup} + \|k_3\|_{\sup},$$

where the inequalities become equalities when k_1 and k_3 are constant functions. Here, we assume that the operator K is a compact and bounded operator (see Appendix 1).

If $g, k_1, \overrightarrow{k_2} \in \mathbb{H}$ are given, then we study the solution of (2.1) to the re-solvent formula

$$S(g, k_1, \overrightarrow{k_2}) := u,$$

which we express as $u = g + \lambda K u$ or as $(\mathcal{I} - \lambda K)u = g$, where

$$\mathcal{I} : \mathcal{C}^{1,v}(D) \to \mathcal{C}^{1,v}(D)$$

denotes the identity operator $\mathcal{I}u = u$ also, we assume that the operator $(\mathcal{I}-\lambda K)^{-1}$ exists. Thus, by using the Fredholm alternative, we have $\|(\mathcal{I} - \lambda K)^{-1}\| < \infty$ and $u = (\mathcal{I} - \lambda K)^{-1}g$.

Therefore we have

$$(Ku)(\mathbf{x}) = \sum_{\mathbf{h}\in\mathbb{Z}^d} \hat{u}(\mathbf{h})e^{2\pi i\mathbf{h}.\mathbf{x}}\big(\hat{k}_1(\mathbf{h}) + \overrightarrow{\hat{k}_2}(\mathbf{h}).2\pi i\mathbf{h}\big)$$

implying

$$\hat{u}(\mathbf{h}) = \hat{g}(\mathbf{h}) + \lambda\big(\hat{k}_1(\mathbf{h}) + \overrightarrow{\hat{k}_2}(\mathbf{h})\cdot 2\pi i\mathbf{h}\big)\hat{u}(\mathbf{h}),$$

hence we conclude the following results:

$$\hat{u}(\mathbf{h}) = \frac{\hat{g}(\mathbf{h})}{1 - \lambda\big(\hat{k}_1(\mathbf{h}) + \overrightarrow{\hat{k}_2}(\mathbf{h}) \cdot 2\pi i \mathbf{h}\big)}, \tag{2.5}$$

and

$$\begin{aligned} \|u\|_{\mathbb{H}} &= \Bigg(\sum_{\mathbf{h}\in\mathbb{Z}^d} \Bigg| \frac{\hat{g}(\mathbf{h})}{1 - \lambda\big(\hat{k}_1(\mathbf{h}) + \overrightarrow{\hat{k}_2}(\mathbf{h}) \cdot 2\pi i \mathbf{h}\big)} \Bigg|^2 r_\alpha(\boldsymbol{\gamma}, \mathbf{h}) \Bigg)^{1/2} \\ &\leq \|(\mathcal{I} - \lambda K)^{-1}\|_{\mathcal{C}} \|g\|_{\mathbb{H}}, \end{aligned} \tag{2.6}$$

where the inequality becomes equality when g, k_1 and $\overrightarrow{k_2}$ are constant functions.

We use QMC-Nyström method by the algorithm

$$A_n(g, k_1, \overrightarrow{k_2}) := u_n \quad \text{or} \quad u_n = g + \lambda K_n u_n.$$

Suppose that

$$\Delta_n := \|(\mathcal{I} - \lambda K)^{-1}\| \|(K - K_n) K_n\| < 1.$$

Then the operator $(\mathcal{I} - \lambda K_n)^{-1}$ exists and

$$\|(\mathcal{I} - \lambda K_n)^{-1}\| \leq \frac{1 + \|(\mathcal{I} - \lambda K)^{-1}\| \|K_n\|}{1 - \Delta_n}.$$

Then u_n is well defined and we have

$$u_n = (\mathcal{I} - \lambda K_n)^{-1} g.$$

Therefore we observe that $\Delta_n < 1$ is essentially as a related condition on the value of n and the equality of the points $\mathbf{t}_1, \ldots, \mathbf{t}_n$.

We assume that $\beta > 0$ and $\mu > 1$ are fixed. Therefore we recall that

$$S(g, k_1, \overrightarrow{k_2}) = (\mathcal{I} - \lambda K)^{-1} g$$

and

$$A_n = A_n(g, k_1, \overrightarrow{k_2}) = (\mathcal{I} - \lambda K_n)^{-1} g.$$

Hence we define the worst case error of a QMC-Nyström method by

$$e_{n,d}(A_n) := \sup_{k_1, \overrightarrow{k_2}, \overrightarrow{k_4}, g \in Æ} \|S(g, k_1, \overrightarrow{k_2}) - A_n(g, k_1, \overrightarrow{k_2})\|_{\sup},$$

where

$$\text{Æ} = \{k_1, \overrightarrow{k_2}, g : \|g\|_{\mathbb{H}} \leq 1,\ \|k_1\|_{\mathbb{H}} \leq \beta,\ \|\overrightarrow{k_2}\|_{\mathbb{H},\infty} \leq \beta,$$
$$\|\nabla k_i\|_{\mathbb{H},\infty} \leq \beta \text{ for } i = 1, 2,\ \|(\mathcal{I} - \lambda K)^{-1}\| \leq \mu\},$$

and

$$\overrightarrow{k_4}(x - y) = \nabla \overrightarrow{k_2}(x - y).\nabla u(t_j).$$

Also, we write the following inequality:

$$\|S(g, k_1, \overrightarrow{k_2}) - A_n(g, k_1, \overrightarrow{k_2})\|_{\sup} \leq e_{n,d}(A_n) \cdot \|g\|_{\mathbb{H}}.$$

Note that the constants β and μ in Æ are mutually independent. We define the initial error associated with the zero algorithm $A_0 \equiv 0$ as

$$e_{0,d}(A_0) := \sup_{k_1, \overrightarrow{k_2}, \overrightarrow{k_4}, g \in \text{Æ}} \|S(g, k_1, \overrightarrow{k_2})\|_{\sup}.$$

For $\varepsilon \in (0, 1)$, we are interested in finding the smallest value of n for which

$$e_{n,d}(A_n) \leq \varepsilon,$$

corresponding to tractability in the absolute sense, or

$$e_{n,d}(A_n) \leq \varepsilon e_{0,d}(A_0),$$

corresponding to tractability in the normalized sense. For $\varepsilon \in (0, 1)$ and $d \geq 1$, we define the following set (see [13, 14]):

$$n^{\text{abs}}(\varepsilon, d) := \min\{n : \exists \text{QMC-Nyström method } A_n \text{ with } e_{n,d}(A_n) \leq \varepsilon\}.$$

The integral equation in this section is said to be QMC-Nyström tractable in the absolute sense iff there exist nonnegative constants C, p and q independent of ε and d such that (see [4])

$$n^{\text{abs}}(\varepsilon, d) \leq C\varepsilon^{-p} d^q, \quad \text{for all } \varepsilon \in (0, 1),\ d \geq 1,$$

and the problem is said to be strongly QMC-Nyström tractable in the absolute sense iff the above condition holds with $q = 0$.

Furthermore, tractability and strong tractability in the normalized sense are defined in a similar way, with $n^{\text{abs}}(\varepsilon, d)$ replaced by

$$n^{\text{nor}}(\varepsilon, d) := \min\{n : \exists \text{QMC-Nyström method } A_n \text{ with } e_{n,d}(A_n) \leq \varepsilon e_{0,d}\}.$$

On the other hand, we obtain the initial error, from (1.6) and (2.6),

$$\|S(g,k_1,\overrightarrow{k_2})\|_{\sup} \leq \|(\mathcal{I}-\lambda K)^{-1}\|\|g\|_{\mathbb{H}} \prod_{j=1}^{d}\big(1+2\zeta(\alpha)\gamma_j\big)^{1/2}$$
$$\leq \mu \prod_{j=1}^{d}\big(1+2\zeta(\alpha)\gamma_j\big)^{1/2},$$

that is an upper bound on the initial error $e_{0,d}$ and does not depend on β.

Now, we obtain a lower bound on initial error. We assume that

$$\kappa_1(\mathbf{x},\mathbf{y}) = c := \min\Big(\beta, \frac{\mu-\lambda L-1}{\mu\lambda}\Big),$$

where

$$L = \int_D \overrightarrow{k_2}\cdot\nabla g d\mathbf{y}$$

and $\overrightarrow{k_2}$ is a constant vector such that $\|\overrightarrow{k_2}\|_{\mathbb{H},\infty} \leq \beta$ and $\|\nabla\overrightarrow{k_2}\|_{\mathbb{H},\infty} \leq \beta$. We define g such that

$$\frac{\hat{g}(\mathbf{h})}{1-\lambda(\hat{k}_1(\mathbf{h})+\hat{\overrightarrow{k_2}}(\mathbf{h}).2\pi i\mathbf{h})} = \frac{1}{Gr_\alpha(\gamma,\mathbf{h})},$$

where

$$G := \prod_{j=1}^{d}\big(1+2\zeta(\alpha)\gamma_j\big)^{1/2}$$

and $\|g\|_{\mathbb{H}} \leq 1$.

On the other hand, if we put $a_i(\mathbf{x}) = 1$, $b_i(\mathbf{y}) = c$, and $k_i(\mathbf{x}-\mathbf{y}) = a_i(\mathbf{x})b_i(\mathbf{y})$, $i = 1, 2$, in the equation

$$u(\mathbf{x}) = g(\mathbf{x}) + \lambda\Big[\int_D k_1(\mathbf{x}-\mathbf{y})u(\mathbf{y})d\mathbf{y} + \int_D \overrightarrow{k_2}(\mathbf{x}-\mathbf{y})\cdot\nabla u(\mathbf{y})d\mathbf{y}\Big], \tag{2.7}$$

then we have

$$u(\mathbf{x}) = g(\mathbf{x}) + \lambda c_1 + \lambda c_2, \tag{2.8}$$

where $c_1 = \int_D cu(\mathbf{y})d\mathbf{y}$ and $c_2 = \int_D \overrightarrow{k_2}.\nabla u(\mathbf{y})d\mathbf{y}$. By differentiating from (2.8), we have

$$\nabla u(\mathbf{x}) = \nabla g(\mathbf{x}). \tag{2.9}$$

Substituting (2.8) and (2.9) in (2.7), we write

$$c_1 = \frac{1}{1-\lambda c}\Big(c\int_D g(\mathbf{y})d\mathbf{y} + \int_D \overrightarrow{k_2}\cdot\nabla g d\mathbf{y} + c_2(\lambda c-1)\Big).$$

Therefore c_2 is a free parameter and c_1 is obtained in terms of c_2; then we have

$$\|(\mathcal{I}-\lambda K)^{-1}\| = \|u\| \leq 1+\frac{\lambda}{1-\lambda c}\big(c+L+c_2(\lambda c-1)\big)+\lambda c_2 = \frac{1+\lambda L}{1-\lambda c} \leq \mu,$$

such that $c \leq \frac{\mu-\lambda L-1}{\mu\lambda}$.

By the above assumptions, (1.6) and (2.5), we know

$$\|S(g,k_1,\overrightarrow{k_2})\|_{\sup} = \left\|\frac{1}{G}\sum_{\mathbf{h}\in\mathbb{Z}^d}\frac{e^{2\pi i\mathbf{h}.\mathbf{x}}}{r_\alpha(\gamma,\mathbf{h})}\right\|_{\sup} = \prod_{j=1}^{d}\big(1+2\zeta(\alpha)\gamma_j\big)^{1/2},$$

and we know that for proving the above inequality, $S(g,k_1,\overrightarrow{k_2})$ is a factor of

$$\sum_{\mathbf{h}\in\mathbb{Z}^d}\frac{e^{2\pi i\mathbf{h}.\mathbf{x}}}{r_\alpha(\gamma,\mathbf{h})}.$$

Then we have

$$\|u\|_{\sup} = \|S(g,k_1,\overrightarrow{k_2})\| = \|u\|_{\mathbb{H}}\prod_{j=1}^{d}\big(1+2\zeta(\alpha)\gamma_j\big)^{1/2}.$$

On the other hand, we have $\|u\|_{\mathbb{H}} = 1$, because

$$\begin{aligned}
\|u\|_{\mathbb{H}} &= \Bigg(\sum_{\mathbf{h}\in\mathbb{Z}^d}|\hat{u}(\mathbf{h})|^2 r_\alpha(\gamma,h)\Bigg)^{1/2}\\
&= \Bigg(\sum_{\mathbf{h}\in\mathbb{Z}^d}\frac{1}{|Gr_\alpha(\gamma,\mathbf{h})|^2}r_\alpha(\gamma,\mathbf{h})\Bigg)^{1/2}\\
&= \Bigg(\frac{1}{G^2}\sum_{\mathbf{h}\in\mathbb{Z}^d}\frac{1}{r_\alpha(\gamma,\mathbf{h})}\Bigg)^{1/2}\\
&= \frac{1}{G}\Bigg(\frac{1}{r_\alpha(\gamma,\mathbf{0})}+\sum_{\mathbf{h}\neq\mathbf{0}}\frac{1}{r_\alpha(\gamma,\mathbf{h})}\Bigg)^{1/2}\\
&= \frac{1}{G}\Bigg(\frac{1}{\prod_{j=1}^{d}1}+\sum_{\mathbf{h}\neq\mathbf{0}}\prod_{j=1}^{d}\gamma_j|\mathbf{h}|^{-\alpha}\Bigg)^{1/2}\\
&= \frac{1}{G}\Bigg(\prod_{j=1}^{d}1+\prod_{j=1}^{d}\gamma_j\cdot 2\sum_{\mathbf{h}\neq\mathbf{0}}|\mathbf{h}|^{-\alpha}\Bigg)^{1/2} = 1.
\end{aligned}$$

In the above, we assume that $g = k_1 = c$ and we show another lower bound for $\|(\mathcal{I}-\lambda K)^{-1}\| = \frac{c}{1-\lambda c}$. Thus, we have the following proposition.

Proposition 2.1. *If* $k_1, \overrightarrow{k_2}, k_3, \overrightarrow{k_4} \in$ Æ, *then*

$$\max\left(\frac{c}{1-\lambda c}, \prod_{j=1}^{d}\big(1+2\zeta(\alpha)\gamma_j\big)^{1/2}\right) \le e_{0,d} \le \mu \prod_{j=1}^{d}\big(1+2\zeta(\alpha)\gamma_j\big)^{1/2}.$$

Hence we have a lower bound and upper bound on the initial error with the same dependence on d; in other words we know exactly how the initial error increases with d.

Proposition 2.2. *If* $c := \min(\beta, \frac{\mu-\lambda L-1}{\mu\lambda})$, *then the worst case error for the QMC-Nyström method* (2.4) *satisfies*

$$e_{n,d}(A_n) \ge \frac{\lambda c}{1-\lambda c}\max\left(\frac{2\zeta(\alpha)\gamma_1}{n^\alpha}, \frac{1}{n}\prod_{j=1}^{d}(1+2\zeta(\alpha)w_\alpha\gamma_j)-1\right)^{1/2},$$

where $w_\alpha \le 1$ *is a constant independent of* n *and* d.

Proof. We consider $k_1 = c := \min(\beta, \frac{\mu-\lambda L-1}{\mu\lambda})$, $\overrightarrow{k_2}$ and $L = \int_D \overrightarrow{k_2}\cdot\nabla g\, d\mathbf{y}$ such that

$$\|\overrightarrow{k_2}\|_{\mathbb{H},\infty} < \beta \quad \text{and} \quad \|\nabla\overrightarrow{k_2}\| \le \beta,$$

hence we write

$$u - u_n = \frac{\lambda}{1-\lambda c}\left[c\left(\int_D g(\mathbf{y})d\mathbf{y} - \frac{1}{n}\sum_{i=1}^{n} g(t_i)\right)\right].$$

Therefore we have the inequality

$$\begin{aligned} e_{n,d}(A_n) &\ge \sup_{\|g\|_{\mathbb{H}}\le 1} \|S(g,k_1,\overrightarrow{k_2}) - A_n(g,k_1,\overrightarrow{k_2})\|_{\sup} \\ &\ge \frac{\lambda}{1-\lambda c}\sup_{\|g\|_{\mathbb{H}}\le 1}\left(c\left|\int_D g(\mathbf{y})d\mathbf{y} - \frac{1}{n}\sum_{i=1}^{n} g(\mathbf{t}_i)\right|\right) \\ &= \frac{\lambda c}{1-\lambda c} e_{n,d}^{\text{wor-int}}(\mathbf{t}_1,\dots,\mathbf{t}_n). \end{aligned}$$

Hence we conclude

$$e_{n,d}^{\text{wor-int}}(\mathbf{t}_1,\dots,\mathbf{t}_n) \ge e_{n,1}^{\text{wor-int}}\left(0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}\right) = \left(\frac{2\zeta(\alpha)\gamma_1}{n^\alpha}\right)^{1/2}, \quad (2.10)$$

where $e_{n,d}^{\text{wor-int}}(\mathbf{t}_1,\dots,\mathbf{t}_n)$ denotes the worst case integration error in $\mathbb{H}$ using quadrature points $\mathbf{t}_1,\dots,\mathbf{t}_n$. If we consider Sharygin's lower bound [4], then this rate of convergence of $\mathcal{O}(n^{-\alpha/2})$ is optimal for the integration problem in weighted

Korobov spaces. In fact, it was proved in [4, 5] that a generating vector $\mathbf{z}$ for a rank-1 lattice rule constructs component-by-component to achieve the rate of convergence $\mathcal{O}(n^{-\alpha/2+\delta})$, $\delta > 0$.

Also, some authors have proved (see [20, 21]) that

$$e_{n,d}^{\text{wor-int}}(\mathbf{t}_1, \ldots, \mathbf{t}_n) \geq \left(\frac{1}{n} \prod_{j=1}^{d} (1 + 2\zeta(\alpha) w_\alpha \gamma_j) - 1 \right)^{1/2}, \tag{2.11}$$

where $w_\alpha := \min(1, 1/(2\gamma_1 |\theta_{\min}|)) \leq 1$, with $-1 < \theta_{\min} < -1 + 2^{-\alpha}$ denoting the minimum of the function

$$\theta(x) = \sum_{h=1}^{\infty} \frac{\cos(2\pi h x)}{h^\alpha}$$

(also, see [15, 16]). □

Moreover, it was proved in [25, 26] that the integration problem in weighted Korobov spaces is strongly QMC tractable iff (1.7) holds, and QMC tractable iff (1.8) holds.

In the following we obtain an upper bound for the worse case error. We recall that

$$(\mathcal{I} - \lambda K_n) u_n = g$$

and

$$(\mathcal{I} - \lambda K_n) u = (\mathcal{I} - \lambda K) u + (\lambda K - \lambda K_n) u = g + \lambda (K - K_n) u.$$

Therefore we obtain

$$u - u_n = \lambda (\mathcal{I} - \lambda K_n)^{-1} (K - K_n) u.$$

Thus, we have the following inequality:

$$\begin{aligned} \|S(g, k_1, \overrightarrow{k_2}) - A_n(g, k_1, \overrightarrow{k_2})\|_{\sup} &= \|u - u_n\|_{\sup} \\ &\leq \lambda \|(\mathcal{I} - \lambda K_n)^{-1}\| \|(K - K_n) u\|_{\sup}. \end{aligned}$$

Therefore we have

$$\|(\mathcal{I} - \lambda K_n)^{-1}\| \leq \frac{1 + \|(\mathcal{I} - \lambda K)^{-1}\| \|K_n\|}{1 - \Delta_n}, \tag{2.12}$$

where

$$\Delta_n := \|(\mathcal{I} - \lambda K)^{-1}\| \|(K - K_n) K_n\| < 1.$$

Therefore we conclude

$$\|K_n\| \le \|k_1\|_{\sup} + \|k_3\|_{\sup} \le (\|k_1\|_{\mathbb{H}} + \|k_3\|_{\mathbb{H}}) \prod_{j=1}^{d} \big(1 + 2\zeta(\alpha)\gamma_j\big)^{1/2}. \quad (2.13)$$

Hence we write

$$\|u - u_n\|_{\sup} \le \frac{1 + \|(\mathcal{I} - \lambda K)^{-1}\| (\|k_1\|_{\mathbb{H}} + \|k_3\|_{\mathbb{H}}) \prod_{j=1}^{d} \big(1 + 2\zeta(\alpha)\gamma_j\big)^{1/2}}{1 - \|(\mathcal{I} - \lambda K)^{-1}\| \|(K - K_n)K_n\|} \times \lambda \|(K - K_n)u\|_{\sup}.$$

The term $\|(K - K_n)K_n\|$ controls whether or not $\Delta_n < 1$ while $\|(K - K_n)u\|_{\sup}$ determines the rate of convergence. It remains to obtain bounds on these two terms.

Let $\mathbf{t}_1, \dots, \mathbf{t}_n$ be rank-1 lattice points generated by $\mathbf{z}$, that is, $\mathbf{t}_i = \{\frac{i\mathbf{z}}{n}\}$ where $\{\mathbf{x}\} = \mathbf{x} - [\mathbf{x}]$. We have

$$\begin{aligned}
((K - K_n)u)(\mathbf{x}) &= \int_D k_1(\mathbf{x} - \mathbf{y}) u(\mathbf{y}) d\mathbf{y} + \int_D \overrightarrow{k_2}(\mathbf{x} - \mathbf{y}) \cdot \nabla u(\mathbf{y}) d\mathbf{y} \\
&\quad - \frac{1}{n} \sum_{i=1}^{n} k_1(\mathbf{x} - \mathbf{t}_i) u(\mathbf{t}_i) - \frac{1}{n} \sum_{i=1}^{n} \overrightarrow{k_2}(\mathbf{x} - \mathbf{t}_i) \cdot \nabla u(\mathbf{t}_i) \\
&= - \sum_{\substack{\mathbf{h} \in \mathbb{Z}^d - \{0\} \\ \mathbf{h} \cdot \mathbf{z} \equiv 0 \ (\mathrm{mod}\ n)}} \hat{U}_{1x}(\mathbf{h}) - \sum_{\substack{\mathbf{h} \in \mathbb{Z}^d - \{0\} \\ \mathbf{h} \cdot \mathbf{z} \equiv 0 \ (\mathrm{mod}\ n)}} \hat{U}_{2\mathbf{x}}(\mathbf{h}),
\end{aligned}$$

where

$$\begin{aligned}
U_{1\mathbf{x}}(\mathbf{y}) &= k_1(\mathbf{x} - \mathbf{y}) u(\mathbf{y}), \\
U_{2\mathbf{x}}(\mathbf{y}) &= \overrightarrow{k_2}(\mathbf{x} - \mathbf{y}) \cdot \nabla u(\mathbf{y}), \\
\hat{U}_{1\mathbf{x}}(\mathbf{h}) &= \int_D k_1(\mathbf{x} - \mathbf{y}) u(\mathbf{y}) e^{-2\pi i \mathbf{h}.\mathbf{y}} d\mathbf{y} \\
&= \int_D \Big(\sum_{\mathbf{l} \in \mathbb{Z}^d} \hat{k}_1(\mathbf{l}) e^{2\pi i \mathbf{l}.(\mathbf{x} - \mathbf{y})} \Big) u(\mathbf{y}) e^{-2\pi i \mathbf{h}.\mathbf{y}} d\mathbf{y} \\
&= \sum_{\mathbf{l} \in \mathbb{Z}^d} \hat{k}_1(\mathbf{l}) e^{2\pi i \mathbf{l}.\mathbf{x}} \int_D u(\mathbf{y}) e^{-2\pi i (\mathbf{h} + \mathbf{l}).\mathbf{y}} d\mathbf{y} \\
&= \sum_{\mathbf{l} \in \mathbb{Z}^d} \hat{k}_1(\mathbf{l}) e^{2\pi i \mathbf{l}.\mathbf{x}} \hat{u}(\mathbf{h} + \mathbf{l}),
\end{aligned}$$

and

$$\begin{aligned}\hat{U}_{2\mathbf{x}}(\mathbf{h}) &= \int_D \overrightarrow{k_2}(\mathbf{x}-\mathbf{y})\cdot\nabla u(\mathbf{y})e^{-2\pi i\mathbf{h}.\mathbf{y}}d\mathbf{y}\\ &= \int_D\Big(\sum_{\mathbf{l}\in\mathbb{Z}^d}\hat{\overrightarrow{k_2}}(\mathbf{l})e^{2\pi i\mathbf{l}.(\mathbf{x}-\mathbf{y})}\Big).\nabla u(\mathbf{y})e^{-2\pi i\mathbf{h}.\mathbf{y}}d\mathbf{y}\\ &= \sum_{\mathbf{l}\in\mathbb{Z}^d}\hat{\overrightarrow{k_2}}(\mathbf{l})e^{2\pi i\mathbf{l}.\mathbf{x}}.\int_D\nabla u(\mathbf{y})e^{-2\pi i(\mathbf{h}+\mathbf{l}).\mathbf{y}}d\mathbf{y}\\ &= \sum_{\mathbf{l}\in\mathbb{Z}^d}\hat{\overrightarrow{k_2}}(\mathbf{l})e^{2\pi i\mathbf{l}.\mathbf{x}}\hat{u}(\mathbf{h}+\mathbf{l})2\pi i.(\mathbf{h}+\mathbf{l}).\end{aligned}$$

Hence we have

$$\begin{aligned}(K-K_n)u = &-\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0\ (\mathrm{mod}\ n)}}\sum_{\mathbf{l}\in\mathbb{Z}^d}\hat{k}_1(\mathbf{l})\hat{u}(\mathbf{h}+\mathbf{l})e^{2\pi i\mathbf{l}.\mathbf{x}}\\ &-\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0\ (\mathrm{mod}\ n)}}\sum_{\mathbf{l}\in\mathbb{Z}^d}\hat{\overrightarrow{k_2}}(\mathbf{l})\hat{u}(\mathbf{h}+\mathbf{l})e^{2\pi i\mathbf{l}.\mathbf{x}}2\pi i.(\mathbf{h}+\mathbf{l}).\end{aligned}$$

Proposition 2.3. *Suppose there exists an integer vector* $\mathbf{z}$ *for* $S_{n,d}(\mathbf{z})$ *defined by*

$$S_{n,d}(\mathbf{z}) = \Bigg(\sum_{\mathbf{l}\in\mathbb{Z}^d}\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0\ (\mathrm{mod}\ n)}}\frac{1}{r_\alpha(\gamma,\mathbf{h}+\mathbf{l})r_\alpha(\gamma,\mathbf{h})}\Bigg)^{1/2} \tag{2.14}$$

such that

$$S_{n,d}(\mathbf{z}) < \frac{1}{4\mu\beta^2}.$$

Then the worst case error for the lattice-Nyström method satisfies

$$e_{n,d}(A_n) \le \frac{\lambda(1+2\mu\beta)\prod_{j=1}^d\big(1+2\zeta(\alpha)\gamma_j\big)^{1/2}}{1-4\mu\beta^2 S_{n,d}(\mathbf{z})}\beta(\mu+1)S_{n,d}(\mathbf{z}).$$

Proof. See Appendix 2. □

In [4, 8, 9, 15, 16, 19], we observe the following CBC algorithm for constructing a generating vector $\mathbf{z}$.

Algorithm 2.4. If n is a prime number, then:

Step 1: Set $z_1 = 1$.

Step 2: For $t = 2, 3, \dots, d$, with $z_1, \dots, z_{t-1}$ already chosen and fixed, find a $z_t \in \{1, 2, \dots, n-1\}$ to minimize $S_{n,t}(z_1, \dots, z_{t-1}, z_t)$.

In this case, the components of the generating vector $\mathbf{z}$ can be restricted to the set $\{1, 2, \dots, n-1\}$. It causes to the optimal rate of convergence $\mathcal{O}(n^{-\alpha/2+\delta})$ for $\delta > 0$.

Proposition 2.5. *If we assume that*

$$\sum_{j=1}^{\infty} \gamma_j^{1/(\alpha-2\delta)} < \infty,$$

and n is a prime number such that

$$n \geq (8\mu\beta^2)^2 2^{6\alpha} \prod_{j=1}^{d} \big(1 + 2(1 + 2^{-3\alpha})^{1/2}\zeta(\alpha)\gamma_j\big)^2, \tag{2.15}$$

then the generating vector $\mathbf{z}^*$ *constructed by Algorithm* 2.4 *completes the optimal rate of convergence, with*

$$e_{n,d}(A_n) \leq C_{d,\delta} n^{-\alpha/2+\delta} \quad \frac{e_{n,d(A_n)}}{e_{0,d}} \leq \tilde{C}_{d,\delta} n^{-\alpha/2+\delta}$$

for all $\delta \in (0, \min(2^{-3\alpha}, (\alpha-1)/2))$*, where* $C_{d,\delta}$ *and* $\tilde{C}_{d,\delta}$ *are independent of* $n = n(\delta, d)$.

Proof. According to [9], we suppose that $\mathbf{z}^* \in \{1, 2, \dots, n-1\}^d$ is constructed by Algorithm 2.4, such that n is a prime number. Then we have

$$S_{n,d}(\mathbf{z}^*) \leq \frac{1}{\delta n^{1/(2p)}} \prod_{j=1}^{d} \big(1 + 2(1 + \delta^p)^{1/2}\zeta(\alpha p)\gamma_j^p\big)^{1/p},$$

for all $p \in (1/\alpha, 1]$ and $\delta \in (0, 2^{-3\alpha}]$. We now obtain a sufficient condition on n to ensure that

$$S_{n,d}(\mathbf{z}) < \frac{1}{4\mu\beta^2}.$$

It is enough to choose n such that the upper bound in the above inequality with $p = 1$ and $\delta = 2^{-3\alpha}$ is not greater than $\frac{1}{8\mu\beta^2}$. In other words, if we write (2.15), then

$$S_{n,d}(\mathbf{z}) \leq \frac{1}{4\mu\beta^2}.$$

If we consider

$$S_{n,d}(\mathbf{z}) \le \frac{1}{\delta n^{1/(2p)}} \prod_{j=1}^{d} \left(1 + 2(1+\delta^p)^{1/2}\zeta(\alpha p)\gamma_j^p\right)^{1/p} \le \frac{1}{8\mu\beta^2},$$

then we have

$$\delta n^{1/(2p)} \ge 8\mu\beta^2 \prod_{j=1}^{d} \left(1 + 2(1+\delta^p)^{1/2}\zeta(\alpha p)\gamma_j^p\right)^{1/p}.$$

Now, we put $p = 1$ and $\delta = 2^{-3\alpha}$; then

$$n \ge (8\mu\beta^2)^2 2^{6\alpha} \prod_{j=1}^{d} \left(1 + 2(1+2^{-3\alpha})^{1/2}\zeta(\alpha)\gamma_j\right)^2,$$

and we conclude from the above propositions that

$$e_{n,d}(A_n) \le \frac{2\lambda(1+2\mu\beta)\beta(\mu+1)}{\delta n^{1/(2p)}} \times \prod_{j=1}^{d} \left(1 + 2(1+\delta^p)^{1/2}\zeta(\alpha p)\gamma_j^p\right)^{1/p} (1+2\zeta(\alpha)\gamma_j)^{1/2},$$

for all $p \in (1/\alpha, 1]$ and $\delta \in (0, 2^{-3\alpha}]$. On the other hand, we know from [4]

$$\prod_{j=1}^{d} (1 + x_j) \le (d+1)^{\sum_{j=1}^{d} \frac{x_j}{\log(d+1)}}, \qquad (2.16)$$

for all $x_j > 0$, we see that the requirement (2.15) on n does not grow with d if (1.7) holds, and it grows only polynomially with d when (1.8) holds. The conditions (1.7) and/or (1.8) are also sufficient to ensure that $e_{n,d}(A_n)$ does not grow faster than polynomially with d. If we assume that $p = 1/(\alpha - 2\delta)$ and $\delta \le \min(2^{-3\alpha}, (\alpha-1)/2)$, then we have

$$e_{n,d}(A_n) = \mathcal{O}(n^{-\alpha/2+\delta})$$

because in the above formula $n^{-1/(2p)} = n^{-(\alpha-2\delta)/2} = n^{-\alpha/2+\delta}$. However, we will need to assume stronger conditions on the weights if we have the optimal rate of convergence at the same time. □

On the other hand, we analyze tractability in the normalized sense. For given $\varepsilon \in (0,1)$, we find the smallest n for which $e_{n,d}(A_n) \le \varepsilon$. We observe that it is sufficient to insist that

$$S_{n,d}(\mathbf{z}) \le \frac{1}{\varepsilon^{-1}\lambda(1+2\mu\beta)\beta(\mu+1)\prod_{j=1}^{d}\left(1+2\zeta(\alpha)\gamma_j\right)^{1/2} + 4\mu\beta^2}, \qquad (2.17)$$

the right-hand side of which is less than $1/(4\mu\beta^2)$. Using Algorithm 2.4, we generate a vector $\mathbf{z}$ satisfying (2.17) such that

$$n \geq \mathrm{pr}\left(\min_{p \in (1/\alpha, 1] \text{ and } \delta \in (0, 2^{-3\alpha}]} \left[\frac{1}{\delta^{2p}} \prod_{j=1}^{d} \big(1 + 2(1+\delta^p)^{1/2} \zeta(\alpha p) \gamma_j^p\big)^2 \times \Bigg(\varepsilon^{-1} \lambda (1 + 2\mu\beta) \beta(\mu+1) \times \prod_{j=1}^{d} (1 + 2\zeta(\alpha)\gamma_j)^{1/2} + 4\mu\beta^2 \Bigg)^{2p} \right] \right), \tag{2.18}$$

where $\mathrm{pr}(x)$ denotes the smallest prime number greater than or equal to x. Hence we conclude that $n^{\mathrm{nor}(\varepsilon,d))}$ is less than or equal to the right-hand side of (2.18). Hence, for tractability in the normalized sense we obtain

$$n^{\mathrm{nor}}(\varepsilon, d) \leq \mathrm{pr}\left(\min_{\lambda \in (1/\alpha, 1] \text{ and } \delta \in (0, 2^{-3\alpha}]} \left[\frac{1}{\delta^2 p} \prod_{j=1}^{d} \big(1 + 2(1+\delta^p)^{1/2} \zeta(\alpha p) \gamma_j^p\big)^2 \times \big(\varepsilon^{-1} \lambda (1 + 2\mu\beta) \beta(\mu+1) + 4\mu\beta^2 \big)^{2p} \right] \right).$$

2.2 Numerical experiments for the first case study

In this section, we present some numerical results for the proposed scheme (2.4) by using the CBC algorithm. We carry out (2.4) by using an AMD Opteron computer with 15 Gigabytes RAM memory with 2.2 GHz CPU for these experiments.

In (2.4), we assume that

$$k(x) = \frac{\prod_{i=1}^{d} e^{-x_i}}{p(x)}, \quad p(x) = \prod_{i=1}^{d} \frac{e^{-x_i^2}}{2\sqrt{2\pi}},$$

$\lambda = 1$, $\partial D_1 = \partial D$, $D = [0,1]^d$ and

$$u(x) = \prod_{i=1}^{d} (x_i e^{-x_i^2} - 1).$$

Hence we obtain $g(x)$ by the following cases. Therefore we will compare exact solution with approximation solution. The evolution of the absolute error of this method, $\|u - u_n\|_\infty$, for $d = 10, 20$ and $n = 107, 523, 1009$ are given in Tables 1–4 based on CPU times. In this case, we will consider two different examples in (2.1).

Example 2.6. We have

$$\begin{cases} a_1(y) = 1, & y \in D, \\ b_0 = 0, & \\ \overrightarrow{a}_2(y) = \overrightarrow{0}, & y \in D. \end{cases}$$

Example 2.7. We have

$$\begin{cases} a_1(y) = 1, & y \in D, \\ \overrightarrow{b}_2 = \overrightarrow{0}, & \\ \overrightarrow{a}_2(y) = \overrightarrow{1}, & y \in D. \end{cases}$$

n	CPU time(s)	$\|u - u_n\|$
107	10.515	0.251e-9
523	139.036	0.103e-8
1009	479.259	0.631e-7

Table 1. $d = 10$ for Example 2.6.

n	CPU time(s)	$\|u - u_n\|$
107	23.675	0.121e-8
523	305.823	0.3213e-8
1009	1123.862	0.321e-7

Table 2. $d = 20$ for Example 2.6.

n	CPU time(s)	$\|u - u_n\|$
107	12.764	0.745e-8
523	175.523	0.194e-8
1009	387.246	0.971e-7

Table 3. $d = 10$ for Example 2.7.

n	CPU time(s)	$\|u - u_n\|$
107	52.876	0.642e-7
523	426.985	0.544-8
1009	1432.765	0.398e-7

Table 4. $d = 20$ for Example 2.7.

3 Second case study of (1.1) or integro-differential equation with convection

In this case, we consider $\alpha_0 = 0$, and $\overrightarrow{\alpha_1}$ as a vector in (1.1); therefore we study the equation

$$p(u) = g(\mathbf{x}) + \lambda_1 \int_D \kappa_1(\mathbf{x}, \mathbf{y}) u(\mathbf{y}) d\mathbf{y} + \lambda_2 \int_D \overrightarrow{\kappa_2}(\mathbf{x}, \mathbf{y}) \cdot \nabla u(\mathbf{y}) d\mathbf{y}, \tag{3.1}$$

with the initial conditions (1.2) and (1.3). Therefore we have

$$\overrightarrow{a_1} \cdot \nabla u(\mathbf{x}) = g(\mathbf{x}) + \lambda_1 \int_D \kappa_1(\mathbf{x}, \mathbf{y}) u(\mathbf{y}) d\mathbf{y} + \lambda_2 \int_D \overrightarrow{\kappa_2}(\mathbf{x}, \mathbf{y}) \cdot \nabla u(\mathbf{y}) d\mathbf{y}. \tag{3.2}$$

We approximate u using the Nyström method based on quasi-Mont-Carlo rules. Let $\mathbf{t}_1, \ldots, \mathbf{t}_n$ be selected points in D that we use to approximate u:

$$\overrightarrow{a_1} \cdot \nabla u_n(\mathbf{x}) = g(\mathbf{x}) + \frac{\lambda_1}{n} \sum_{i=1}^{n} \kappa_1(\mathbf{x}, \mathbf{t}_i) u_n(\mathbf{t}_i) + \frac{\lambda_2}{n} \sum_{i=1}^{n} \overrightarrow{\kappa_2}(\mathbf{x}, \mathbf{t}_i) \cdot \nabla u_n(\mathbf{t}_i)$$

by integrating the above equation and we assume that there is a suitable vector $\overrightarrow{a_1}$ such that

$$u(\mathbf{x}) = \int \overrightarrow{a_1}.\nabla u(\mathbf{x}) d\mathbf{x};$$

therefore we have

$$u_n(\mathbf{x}) = \int g(\mathbf{x}) d\mathbf{x} + \frac{\lambda_1}{n} \sum_{i=1}^{n} \int \kappa_1(\mathbf{x}, \mathbf{t}_i) u_n(\mathbf{t}_i) d\mathbf{x} + \frac{\lambda_2}{n} \sum_{i=1}^{n} \int \overrightarrow{\kappa_2}(\mathbf{x}, \mathbf{t}_i) \cdot \nabla u_n(\mathbf{t}_i) d\mathbf{x}.$$

Here, we obtain $u_n(\mathbf{t}_1), \ldots, u_n(\mathbf{t}_n)$, $\nabla u_n(\mathbf{t}_1), \ldots, \nabla u_n(\mathbf{t}_n)$ by solving the following linear system:

$$\begin{cases} \overrightarrow{a_1} \cdot \nabla u_n(\mathbf{t}_j) = g(\mathbf{t}_j) + \dfrac{\lambda_1}{n} \displaystyle\sum_{i=1}^{n} \kappa_1(\mathbf{t}_j, \mathbf{t}_i) u_n(\mathbf{t}_i) \\ \qquad\qquad + \dfrac{\lambda_2}{n} \displaystyle\sum_{i=1}^{n} \overrightarrow{\kappa_2}(\mathbf{t}_j, \mathbf{t}_i) \cdot \nabla u_n(\mathbf{t}_i), \\ u_n(\mathbf{t}_j) = \left(\displaystyle\int g(\mathbf{x}) d\mathbf{x} \right)_{\mathbf{t}_j} + \dfrac{\lambda_1}{n} \left(\displaystyle\sum_{i=1}^{n} \int \kappa_1(\mathbf{x}, \mathbf{t}_i) u_n(\mathbf{t}_i) dx \right)_{\mathbf{t}_j} \\ \qquad\qquad + \dfrac{\lambda_2}{n} \left(\displaystyle\sum_{i=1}^{n} \int \kappa_2(\mathbf{x}, \mathbf{t}_i) \cdot \nabla u_n(\mathbf{t}_i) d\mathbf{x} \right)_{\mathbf{t}_j}, \end{cases} \tag{3.3}$$

where $j = 1, \ldots, n$, and therefore we have $2n$ equations and $2n$ unknowns. We define the integral operator $\widetilde{K} : \mathcal{C}^{1,v}(D) \to \mathcal{C}^{1,v}(D)$ by

$$\widetilde{K}u = \lambda_1 \int_D \kappa_1(\mathbf{x}, \mathbf{y}) u(\mathbf{y}) d\mathbf{y} + \lambda_2(\mathbf{x}, \mathbf{y}) \int_D \overrightarrow{\kappa_2} \cdot \nabla u(\mathbf{y}) d\mathbf{y}, \tag{3.4}$$

and we consider

$$\|\widetilde{K}\| = \lambda_1 \max_{\mathbf{x} \in D} \int_D |k_1| d\mathbf{y} + \lambda_2 \max_{\mathbf{x} \in D} \int_D |k_3| d\mathbf{y}, \tag{3.5}$$

such that $k_1(\mathbf{x} - \mathbf{y}) = \kappa_1(\mathbf{x}, \mathbf{y})$, $\overrightarrow{k_2}(\mathbf{x} - \mathbf{y}) = \overrightarrow{\kappa_2}(\mathbf{x}, \mathbf{y})$ and $k_3 = \overrightarrow{\kappa_2} \cdot \nabla u(\mathbf{y})$. The corresponding discrete operator $\widetilde{K}_n : \mathcal{C}^{1,v}(D) \to \mathcal{C}^{1,v}(D)$ is given by

$$\widetilde{K}_n u = \frac{\lambda_1}{n} \sum_{i=1}^{n} k_1(\mathbf{x} - \mathbf{t}_i) u_n(\mathbf{t}_i) + \frac{\lambda_2}{n} \sum_{i=1}^{n} \overrightarrow{k_2}(\mathbf{x} - \mathbf{t}_i) \cdot \nabla u_n(\mathbf{t}_i), \tag{3.6}$$

with

$$\|\widetilde{K}_n\| = \max_{\mathbf{x} \in D} \frac{\lambda_1}{n} \sum_{i=1}^{n} |k_1(\mathbf{x} - \mathbf{t}_i)| + \max_{\mathbf{x} \in D} \frac{\lambda_2}{n} \sum_{i=1}^{n} |k_3(\mathbf{x} - \mathbf{t}_i)|, \tag{3.7}$$

where $\mathbf{t}_1, \ldots, \mathbf{t}_n \in D$. Thus the above kernels in (3.7) are convolution kernels with periodic form. Therefore we define

$$\|\widetilde{K}\| = \lambda_1 \int_D |k_1(\mathbf{y})| d\mathbf{y} + \lambda_2 \int_D |k_3(\mathbf{y})| dy \le \lambda_1 \|k_1\|_{\sup} + \lambda_2 \|k_3\|_{\sup}, \tag{3.8}$$

and we write

$$\|\widetilde{K}_n\| = \lambda_1 \max_{\mathbf{x}\in D} \frac{1}{n}\sum_{i=1}^{n} |k_1(\mathbf{x}-\mathbf{t}_i)| + \lambda_2 \max_{x\in D} \frac{1}{n}\sum_{i=1}^{n} |k_3(\mathbf{x}-\mathbf{t}_i)|.$$

Therefore we conclude that

$$\|\widetilde{K_n}\| \le \lambda_1\|k_1\|_{\sup} + \lambda_2\|k_3\|_{\sup}. \tag{3.9}$$

3.1 Lower and upper bounds on the worst case error for (3.1)

We assume that $g, k_1, \overrightarrow{k_2} \in \mathbb{H}$ are given functions and we study the solutions of

$$S(g, k_1, \overrightarrow{k_2}) = u$$

from (3.2) as

$$\overrightarrow{a_1}\cdot\nabla u = g + \widetilde{K}u, \tag{3.10}$$

where $\widetilde{K}u = \overrightarrow{a_1}\cdot\nabla u - g$. Hence we have

$$u = \widetilde{K}^{-1}(\overrightarrow{a_1}\cdot\nabla u - g), \tag{3.11}$$

since $\kappa(\mathbf{x},\mathbf{y}) = k(\mathbf{x}-\mathbf{y})$.

On the other hand, we write

$$\begin{aligned}\widetilde{K}u(\mathbf{x}) &= \lambda_1\int_D k_1(\mathbf{x}-\mathbf{y})u(\mathbf{y})d\mathbf{y} + \lambda_2\int_D \overrightarrow{k_2}(\mathbf{x}-\mathbf{y})\cdot\nabla u(\mathbf{y})d\mathbf{y}\\ &= \lambda_1\int_D\Big(\sum_{\mathbf{h}\in\mathbb{Z}^d}\hat{k}_1(\mathbf{h})e^{2\pi i\mathbf{h}.(\mathbf{x}-\mathbf{y})}\Big)\Big(\sum_{\mathbf{h}\in\mathbb{Z}^d}\hat{u}(\mathbf{h})e^{2\pi i\mathbf{h}.\mathbf{y}}d\mathbf{y}\Big)\\ &\qquad + \lambda_2\int_D\Big(\sum_{\mathbf{h}\in\mathbb{Z}^d}\overrightarrow{\hat{k}_2}(\mathbf{h})e^{2\pi i\mathbf{h}.(\mathbf{x}-\mathbf{y})}\Big)\Big(\sum_{\mathbf{h}\in\mathbb{Z}^d}\hat{u}(\mathbf{h})2\pi i\mathbf{h}.e^{2\pi i\mathbf{h}.\mathbf{y}}d\mathbf{y}\Big)\\ &= \lambda_1\sum_{\mathbf{h}\in\mathbb{Z}^d}\hat{k}_1(\mathbf{h})\hat{u}(\mathbf{h})e^{2\pi i\mathbf{h}.\mathbf{x}} + \lambda_2\sum_{\mathbf{h}\in\mathbb{Z}^d}\overrightarrow{\hat{k}_2}(\mathbf{h})\hat{u}(\mathbf{h}).2\pi i\mathbf{h}e^{2\pi i\mathbf{h}.\mathbf{x}}.\end{aligned}$$

Therefore we have

$$\widetilde{K}u(\mathbf{x}) = \sum_{\mathbf{h}\in\mathbb{Z}^d}\hat{u}(\mathbf{h})e^{2\pi i\mathbf{h}.\mathbf{x}}\big(\lambda_1\hat{k}_1(\mathbf{h}) + \lambda_2\overrightarrow{\hat{k}_2}(\mathbf{h})\cdot 2\pi i\mathbf{h}\big). \tag{3.12}$$

Hence, if we put

$$v(\mathbf{h}) = \overrightarrow{a_1}\cdot\nabla u(\mathbf{h}),$$

then we have

$$\hat{v}(\mathbf{h}) = \hat{g}(\mathbf{h}) + \hat{u}(\mathbf{h})\big(\lambda_1\hat{k}_1(\mathbf{h}) + \lambda_2\overrightarrow{\hat{k}_2}(\mathbf{h})\cdot 2\pi i\mathbf{h}\big),$$

or we write

$$\hat{u}(\mathbf{h}) = \frac{\hat{v}(\mathbf{h}) - \hat{g}(\mathbf{h})}{\lambda_1 \hat{k}_1(\mathbf{h}) + \lambda_2 \overrightarrow{\hat{k}_2}(\mathbf{h}) \cdot 2\pi i \mathbf{h}}. \tag{3.13}$$

Thus we conclude that

$$\begin{aligned} \|u\|_{\mathbb{H}} &= \left(\sum_{\mathbf{h} \in \mathbb{Z}^d} \left| \frac{\hat{v}(\mathbf{h}) - \hat{g}(\mathbf{h})}{\lambda_1 \hat{k}_1(\mathbf{h}) + \lambda_2 \overrightarrow{\hat{k}_2}(\mathbf{h}) \cdot 2\pi i \mathbf{h}} \right|^2 r_\alpha(\boldsymbol{\gamma}, \mathbf{h}) \right)^{1/2} \\ &\leq \|\widetilde{K}^{-1}\| \cdot \|\overrightarrow{a_1} \cdot \nabla u - g\|_{\mathbb{H}}. \end{aligned} \tag{3.14}$$

We use QMC-Nyström method by the algorithm $A_n(g, k_1, \overrightarrow{k_2}) := u_n$ or

$$u_n = \widetilde{K}_n^{-1}(\overrightarrow{a_1} \cdot \nabla u - g). \tag{3.15}$$

Also, if we assume that

$$\widetilde{\Delta}_n := \|\widetilde{K}^{-1}\| \|(\widetilde{K}_n - \widetilde{K})\widetilde{K}_n\| < 1, \tag{3.16}$$

then the operator $\widetilde{K}_n^{-1}$ exists and we have

$$\|\widetilde{K}^{-1}\| \leq \frac{1 + \|\widetilde{K}^{-1}\| \|\widetilde{K}_n\|}{1 - \widetilde{\Delta}_n}. \tag{3.17}$$

Now, we consider $\beta > 0$, $\mu > 1$,

$$u = S(g, k_1, \overrightarrow{k_2}) = \widetilde{K}^{-1}(\overrightarrow{a_1} \cdot \nabla u - g)$$

and

$$u_n = A_n(g, k_1, \overrightarrow{k_2}) = \widetilde{K}_n^{-1}(\overrightarrow{a_1} \cdot \nabla u - g).$$

Also, we assume that the worst case error of the QMC-Nyström method is introduced by

$$e_{n,d}(A_n) := \sup_{k_1, \overrightarrow{k_2}, g \in \widetilde{Æ}} \|S(g, k_1, \overrightarrow{k_2}) - A_n(g, k_1, \overrightarrow{k_2})\|_{\sup}, \tag{3.18}$$

where

$$\widetilde{Æ} = \left\{ g, k_1, \overrightarrow{k_2} : \|\overrightarrow{a_1} \cdot \nabla u - g\|_{\mathbb{H}} \leq 1,\ \|k_1\|_{\mathbb{H}}, \|\overrightarrow{k_2}\|_{\mathbb{H},\infty} \leq \beta,\ \|\widetilde{K}^{-1}\| \leq \mu \right\}.$$

Thus, if $g \in \mathbb{H}$ is a linear function, then for all $k_1, \overrightarrow{k_2}$ we have

$$\|S(g, k_1, \overrightarrow{k_2}) - A_n(g, k_1, \overrightarrow{k_2})\|_{\sup} \leq e_{n,d}(A_n) \|g\|_{\mathbb{H}}. \tag{3.19}$$

Moreover, we define the initial error associated with the zero algorithm $A_0 \equiv 0$ as follows:

$$e_{0,d} := \sup_{k_1, \overrightarrow{k_2}, g \in \widetilde{Æ}} \|S(g, k, k_1)\|_{\sup}. \tag{3.20}$$

Therefore, the initial error follows from (1.6) and (3.15) is

$$\|S(g, k_1, k_2)\|_{\sup} \leq \|\widetilde{K}^{-1}\| \|\overrightarrow{a_1} \cdot \nabla u - g\|_{\mathbb{H}} \prod_{j=1}^{d} \big(1 + 2\zeta(\alpha)\gamma_j\big)^{1/2},$$

thus we write

$$\forall g, k_1, \overrightarrow{k_2}, \in Æ, \quad \|S(g, k_1, k_2)\|_{\sup} \leq \mu \prod_{j=1}^{d} \big(1 + 2\zeta(\alpha)\gamma_j\big)^{1/2}. \tag{3.21}$$

Now, we obtain a lower bound on the initial error. Let

$$k_1(x - y) = k_1, \quad \overrightarrow{k_2}(x - y) = \overrightarrow{\alpha_1}$$

where k_1 is a constant so that $\|k_1\|_{\mathbb{H}} = k_1 \leq \beta$ and we assume $\|\overrightarrow{\alpha_1}\|_{\mathbb{H},\infty} \leq \beta$ and $\|\widetilde{K}^{-1}\| \leq \mu$. Now $g(x)$ is defined such that $\|\overrightarrow{a_1} \cdot \nabla u - g\|_{\mathbb{H}} \leq 1$ and

$$\hat{u}(\mathbf{h}) = \frac{\overrightarrow{a_1} \cdot \nabla u(h) - \hat{g}(h)}{\lambda_1 \hat{k}_1(\mathbf{h}) + \lambda_2 \hat{\overrightarrow{k_2}}(\mathbf{h}) \cdot 2\pi i \mathbf{h}} = \frac{1}{G r_\alpha(\boldsymbol{\gamma}, \mathbf{h})}, \tag{3.22}$$

thus for this choice of g, k_1 and $\overrightarrow{k_2}$, we have

$$\|S(g, k_1, \overrightarrow{k_2})\|_{\sup} = \left\| \frac{1}{G} \sum_{\mathbf{h} \in \mathbb{Z}^d} \frac{e^{2\pi i \mathbf{h}.x}}{r_\alpha(\boldsymbol{\gamma}, \mathbf{h})} \right\|_{\sup} = \prod_{j=1}^{d} \big(1 + 2\zeta(\alpha)\gamma_j\big)^{1/2}. \tag{3.23}$$

Hence we conclude the following proposition for the initial error based on the before section.

Proposition 3.1. *Let* $k_1(\mathbf{x} - \mathbf{y}) = k_1$ *and* $\overrightarrow{k_2}(\mathbf{x} - \mathbf{y}) = \overrightarrow{\alpha_1}$ *where*

$$\|k_1\|_{\mathbb{H}} \leq \beta, \quad \|\overrightarrow{\alpha_1}\|_{\mathbb{H}} = \|\overrightarrow{k_2}\|_{\mathbb{H}} \leq \beta \quad \textit{and} \quad \|\widetilde{K}^{-1}\| \leq \mu.$$

Then

$$\prod_{j=1}^{d} \big(1 + 2\zeta(\alpha)\gamma_j\big)^{1/2} \leq e_{0,d} \leq \mu \prod_{j=1}^{d} \big(1 + 2\zeta(\alpha)\gamma_j\big)^{1/2}.$$

Proof. See Appendix 3. □

Proposition 3.2. *Suppose there exists an integer vector* $\mathbf{z}$ *for which* $S_{n,d}(\mathbf{z})$ *is defined in* (2.14) *and we assume that*

$$S_{n,d}(\mathbf{z}) < \frac{1}{\mu(\lambda_1+\lambda_2)^2\beta^2}.$$

Then the worst case error for the lattice-Nyström method satisfies

$$e_{n,d}(A_n) \le \frac{(\lambda_1\mu+\lambda_2)\beta\big[1+\big(\mu\beta(\lambda_1+\lambda_2)\big)\prod_{j=1}^{d}\big(1+2\zeta(\alpha)\gamma_j\big)\big]}{1-\mu\beta^2(\lambda_1+\lambda_2)^2 S_{n,d}(\lambda_1\mu+\lambda_2\beta)} S_{n,d}(\mathbf{z}),$$

and we conclude $e_{n,d}(A_n) = \mathcal{O}(n^{\frac{-\alpha}{2}+\delta})$.

Proof. We obtain a sufficient condition on n to ensure that

$$S_{n,d}(\mathbf{z}) < \frac{1}{\mu\beta^2(\lambda_1+\lambda_2)^2}.$$

It is enough to choose n such that the upper bound in Section 2 with $p=1$ and $\delta = 2^{-3\alpha}$ is not greater than $\frac{1}{2\mu\beta^2(\lambda_1+\lambda_2)^2}$. In other words, if

$$n \ge \big(2\mu(\lambda_1+\lambda_2)^2\beta^2\big)^2 2^{6\alpha}\prod_{j=1}^{d}\big(1+2(1+2^{-3\alpha})^{1/2}\zeta(\alpha)\gamma_j\big)^2, \tag{3.24}$$

then

$$S_{n,d}(\mathbf{z}) \le \frac{1}{2\mu\beta^2(\lambda_1+\lambda_2)^2}$$

because

$$S_{n,d}(\mathbf{z}) \le \frac{1}{\delta n^{1/(2p)}}\prod_{j=1}^{d}\big(1+2(1+\delta^p)^{1/2}\zeta(\alpha p)\gamma_j^p\big)^{1/p} \le \frac{1}{2\mu\beta^2(\lambda_1+\lambda_2)^2}.$$

Therefore we have the following inequality:

$$\delta n^{1/(2p)} \ge 2\mu\beta^2(\lambda_1+\lambda_2)^2\prod_{j=1}^{d}\big(1+2(1+\delta^p)^{1/2}\zeta(\alpha p)\gamma_j^p\big)^{1/p}.$$

Moreover, by Propositions 2.5 and 3.2, we conclude

$$e_{n,d}(A_n) \le \frac{2(1+\mu\beta(\lambda_1+\lambda_2))}{\delta n^{1/(2p)}}\prod_{j=1}^{d}(1+2\zeta(\alpha)\gamma_j)(\lambda_1\mu+\lambda_2)\beta$$
$$\times\prod_{j=1}^{d}\big(1+2(1+\delta^p)^{1/2}\zeta(\alpha p)\gamma_j^p\big)^{1/p}.$$

On the other hand, if we put

$$S_{n,d}(\mathbf{z}) < \frac{1}{2\mu(\lambda_1+\lambda_2)^2\beta^2},$$

then

$$\mu\beta^2(\lambda_1+\lambda_2)^2 S_{n,d} \le \frac{\mu\beta^2(\lambda_1+\lambda_2)^2}{2\mu\beta^2(\lambda_1+\lambda_2)^2} = \frac{1}{2}.$$

Therefore we have the following inequality:

$$\frac{1}{1-\mu\beta^2(\lambda_1+\lambda_2)^2 S_{n,d}(z)} \le 2 \quad \text{for all } p \in (\alpha^{-1}, 1] \text{ and } \delta \in (0, 2^{-3\alpha}].$$

Also, if we put $p = 1/(\alpha - 2\delta)$ with $\delta \le \min(2^{-3\alpha}, (\alpha-1)/2)$, then we obtain

$$e_{n,d}(A_n) = \mathcal{O}(n^{-\alpha/2+\delta})$$

because in the above formula we have $n^{-1/(2p)} = n^{-(\alpha-2\delta)/2} = n^{-\alpha/2+\delta}$. □

Using the property

$$\prod_{j=1}^{d}(1+x_j) \le (d+1)^{\sum_{j=1}^{d}\frac{x_j}{\log(d+1)}} \quad \text{for all } x_j > 0,$$

we see that the requirement (3.24) on n does not grow with d if $\sum_{j=1}^{\infty}\gamma_j < \infty$ holds, and it grows only polynomially with d when

$$\lim_{d\to\infty}\sup \frac{\sum_{j=1}^{d}\gamma_j}{\log(d+1)} < \infty$$

holds.

Proposition 3.3. *Suppose n is a prime number satisfying* (3.24). *Then the generating vector $\mathbf{z}^*$ is constructed by Algorithm* 2.4, *so it achieves the optimal rate of convergence, with*

$$e_{n,d}(A_n) \le C_{d,\delta} n^{-\alpha/2+\delta} \quad \text{and} \quad \frac{e_{n,d}(A_n)}{e_{0,d}} \le \tilde{C}_{d,\delta} n^{-\alpha/2+\delta},$$

for all $\delta \in (0, \min(2^{-3\alpha}, (\alpha-1)/2))$, where $C_{d,\delta}$ and $\tilde{C}_{d,\delta}$ are independent of n but depend on δ and d additionally if we write

$$\sum_{j=1}^{\infty}\gamma_j^{1/(\alpha-2\delta)} < \infty,$$

then the numbers $C_{d,\delta}$, $\tilde{C}_{d,\delta}$ and the requirement (3.24) *on n are bounded independently of d.*

On the other hand, for tractability in the absolute sense, we find the smallest n for $e_{n,d}(A_n) \le \varepsilon$. From the before proposition, we see that it is sufficient to insist that

$$S_{n,d}(\mathbf{z}) \le \frac{1}{\varepsilon^{-1}(\lambda_1\mu+\lambda_2)\beta(1+\mu\beta(\lambda_1+\lambda_2))} \times \frac{1}{\prod_{j=1}^{d}(1+2\zeta(\alpha)\gamma_j)\mu\beta^2(\lambda_1+\lambda_2)^2}, \tag{3.25}$$

the right-hand side of which is less than $1/\mu\beta^2(\lambda_1+\lambda_2)^2$. Using Proposition 2.5, we observe that Algorithm 2.4 will generate a vector $\mathbf{z}$ satisfying (3.25) if we demand that

$$n \ge \mathrm{pr}\Bigg(\min_{p\in(1/\alpha,1] \text{ and } \delta\in(0,2^{-3\alpha}]}\Bigg[\frac{1}{\delta^{2p}}\prod_{j=1}^{d}\big(1+2(1+\delta^p)^{1/2}\zeta(\alpha p)\gamma_j^p\big)^2 \tag{3.26}$$

$$\times\Bigg(\varepsilon^{-1}(\lambda_1\mu+\lambda_2)\beta(1+\mu\beta(\lambda_1+\lambda_2)) \times \prod_{j=1}^{d}(1+2\zeta(\alpha)\gamma_j)+\mu\beta^2(\lambda_1+\lambda_2)^2\Bigg)^{2p}\Bigg]\Bigg), \tag{3.27}$$

where $\mathrm{pr}(x)$ denotes the smallest prime number greater than or equal to x. Hence we conclude that $n^{\mathrm{abs}}(\varepsilon,d)$ is less than or equal to the right-hand side of (3.27).

Similarly, for tractability in the normalized sense we obtain

$$n^{\mathrm{nor}}(\varepsilon,d) \le \mathrm{pr}\Bigg(\min_{p\in(1/\alpha,1] \text{ and } \delta\in(0,2^{-3\alpha}]}\Bigg[\frac{1}{\delta^2 p}\prod_{j=1}^{d}\big(1+2(1+\delta^p)^{1/2}\zeta(\alpha p)\gamma_j^p\big)^2 \times\big(\varepsilon^{-1}(\lambda_1\mu+\lambda_2)\beta(1+\mu\beta(\lambda_1+\lambda_2))+\mu\beta^2(\lambda_1+\lambda_2)^2\big)^{2p}\Bigg]\Bigg).$$

3.2 Numerical experiments for the second case study

In this section, we present some numerical results for the proposed scheme (3.3) by using the CBC algorithm. We carry out (3.3) by using an AMD Opteron computer with 15 Gigabytes RAM memory with 2.2 GHz CPU for these experiments.

In (3.3), we assume that

$$\kappa_1(x) = \kappa_2(x) = \frac{\prod_{i=1}^{d} e^{-x_i}}{p(x)}, \quad p(x) = \prod_{i=1}^{d}\frac{e^{-x_i^2}}{2\sqrt{2\pi}},$$

$\lambda_1 = \lambda_2 = 1$, $\partial D_1 = \partial D$, $D = [0,1]^d$ and

$$u(x) = \prod_{i=1}^{d} (x_i e^{-x_i^2} - 1).$$

Hence we obtain $g(x)$ by the following cases. Therefore we will compare exact solution with approximation solution. The evolution of the absolute error of this method, $\|u - u_n\|_\infty$, for $d = 10, 20$ and $n = 107, 523, 1009$ are given in Tables 5–8 based on CPU times. In this case, we will consider two different examples in (3.3).

Example 3.4. We have

$$\begin{cases} \overrightarrow{a}_1(y) = \overrightarrow{1}, & y \in D, \\ \quad b_0 = 0, \end{cases}$$

Example 3.5. We have

$$\begin{cases} \overrightarrow{a}_1(y) = \overrightarrow{1}, & y \in D, \\ \quad \overrightarrow{b}_2 = \overrightarrow{0}. \end{cases}$$

n	CPU time(s)	$\|u - u_n\|$
107	238.515	0.643e-9
523	743.036	0.543e-8
1009	1654.259	0.687e-6

Table 5. $d = 10$ for Example 3.4.

n	CPU time(s)	$\|u - u_n\|$
107	197.695	0.121e-8
523	875.985	0.459e-9
1009	1487.092	0.043e-7

Table 6. $d = 20$ for Example 3.4.

n	CPU time(s)	$\|u - u_n\|$
107	226.987	0.942e-7
523	654.841	0.044e-8
1009	1304.765	0.654e-6

Table 7. $d = 10$ for Example 3.5.

n	CPU time(s)	$\|u - u_n\|$
107	152.876	0.642e-7
523	426.985	0.544-8
1009	1752.765	0.398e-7

Table 8. $d = 20$ for Example 3.5.

A Appendix 1

Consider the following sequence of linear operators:

$$K_n : \mathcal{C}^{1,v}(D) \to \mathcal{P}_n$$

given by

$$K_n u = \frac{1}{n}\sum_{i=1}^{n} k_1(\mathbf{x} - \mathbf{t}_i)u(\mathbf{t}_i) + \frac{1}{n}\sum_{i=1}^{n} \overrightarrow{k_2}(\mathbf{x} - \mathbf{t}_i) \cdot \nabla u(\mathbf{t}_i),$$

where $\mathcal{P}_n$ is the space of polynomials of degree less than or equal to n in D. Therefore the dimension of this space is finite. On the other hand, we assume that the kernel functions satisfy the following conditions:

$$\max_{x \in D} \sum_{i=1}^{n} |k_1(\mathbf{x} - \mathbf{t}_i)| = B_{1,n} < \infty$$

and

$$\max_{\mathbf{x} \in D} \sum_{i=1}^{n} \|\overrightarrow{k_2}(\mathbf{x} - \mathbf{t}_i)\| = B_{2,n} < \infty.$$

Also, we consider the following linear operator:

$$K : \mathcal{C}^{1,v}(D) \to \mathcal{C}^{1,v}(D)$$

given by

$$Ku = \int_D k_1(\mathbf{x}-\mathbf{t})u(\mathbf{t})dt + \int_D \overrightarrow{k_2}(\mathbf{x}-\mathbf{t}) \cdot \nabla u(\mathbf{t})dt.$$

Theorem A.1. *Each operator $K_n : \mathcal{C}^{1,v}(D) \to \mathcal{P}_n$ has the following properties:*

(1) *it is a bounded operator,*

(2) $\dim K_n(\mathcal{P}_n) < \infty$,

(3) *the operator K_n is compact.*

Proof. It is clear that we can prove that K_n is a sequence of bounded operators (see [1, Chapter 12]) and according to the theorem of *finite dimensional domain or rang* (see [2, Chapter 8]), the proof is completed. □

Theorem A.2 (Sequence of compact linear operators). *Let $\{K_n\}$ be a sequence of compact linear operators from the normed space $\mathcal{C}^{1,v}(D)$ into the Banach space $\mathcal{P}_n$. If $\{K_n\}$ is uniformly operator convergent, say, $\|K_n - K\| \to 0$, then the limit operator K is compact.*

Proof. See [2]. □

Theorem A.3 (Continuity). *Let X and Y be normed spaces. Then every compact linear operator $K : X \to Y$ is bounded, hence continuous.*

Proof. See [2]. □

B Appendix 2. Proof of Proposition 2.3

It follows from the Cauchy–Schwarz inequality that

$$\begin{aligned}\|(K-K_n)u\|_{\sup} &= \sup_{\mathbf{x}\in D}\Bigg|\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \hat{U}_{1\mathbf{x}}(\mathbf{h}) + \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \hat{U}_{2\mathbf{x}}(\mathbf{h})\Bigg| \\ &= \sup_{\mathbf{x}\in D}\Bigg|\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \hat{k}_1(\mathbf{l})\hat{u}(\mathbf{h}+l)e^{2\pi i\mathbf{l}.\mathbf{x}} \\ &\qquad + \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \overrightarrow{\hat{k}_2}(\mathbf{l})\hat{u}(\mathbf{h}+\mathbf{l})e^{2\pi i\mathbf{l}.\mathbf{x}} 2\pi i(\mathbf{h}+\mathbf{l})\Bigg|\end{aligned}$$

$$\le \overbrace{\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod{n}}} \sum_{\mathbf{l}\in\mathbb{Z}^d} |\hat{k}_1(\mathbf{l})||\hat{u}(\mathbf{h}+\mathbf{l})|}^{=I}$$

$$+ \overbrace{\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod{n}}} \sum_{\mathbf{l}\in Z^d} \left|\overrightarrow{\hat{k}_2}(l)\hat{u}(\mathbf{h}+\mathbf{l})2\pi i(\mathbf{h}+\mathbf{l})\right|}^{=II}.$$

We have

$$\begin{aligned} I &= \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod{n}}} \sum_{\mathbf{l}\in\mathbb{Z}^d} |\hat{k}_1(\mathbf{l})||\hat{u}(\mathbf{h}+\mathbf{l})| \\ &\le \sum_{\mathbf{l}\in\mathbb{Z}^d}\Bigg[|\hat{k}_1(\mathbf{l})|\Bigg(\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod{n}}} |\hat{u}(\mathbf{h}+\mathbf{l})|^2 r_\alpha(\gamma,\mathbf{h}+\mathbf{l})\Bigg)^{1/2} \\ &\qquad \times\Bigg(\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod{n}}} \frac{1}{r_\alpha(\gamma,\mathbf{h}+\mathbf{l})}\Bigg)^{1/2}\Bigg], \end{aligned}$$

by the Cauchy–Schwarz inequality

$$\begin{aligned} &\le \Bigg(\sum_{\mathbf{l}\in\mathbb{Z}^d} |\hat{k}_1(\mathbf{l})|^2 \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod{n}}} |\hat{u}(\mathbf{h}+\mathbf{l})|^2 r_\alpha(\gamma,\mathbf{h}+\mathbf{l})\Bigg)^{1/2} \\ &\qquad \times\Bigg(\sum_{\mathbf{l}\in\mathbb{Z}^d}\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod{n}}} \frac{1}{r_\alpha(\gamma,h+l)}\Bigg)^{1/2} \\ &= \Bigg(\sum_{l\in\mathbb{Z}^d} |\hat{k}_1(\mathbf{l})|\|u\|_{\mathbb{H}}\Bigg)\Bigg(\sum_{\mathbf{l}\in\mathbb{Z}^d}\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod{n}}} \frac{1}{r_\alpha(\gamma,\mathbf{h}+\mathbf{l})}\Bigg)^{1/2} \\ &= \|u\|_{\mathbb{H}}\Bigg(\sum_{\mathbf{l}\in\mathbb{Z}^d} |\hat{k}_1(\mathbf{l})|^2 r_\alpha(\gamma,\mathbf{h})^{1/2}\Bigg) \\ &\qquad \times\Bigg(\sum_{\mathbf{l}\in\mathbb{Z}^d}\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod{n}}} \frac{1}{r_\alpha(\gamma,\mathbf{h}+\mathbf{l})r_\alpha(\gamma,h)}\Bigg)^{1/2} \end{aligned}$$

$$
\begin{aligned}
&= \|u\|_{\mathbb{H}} \|k_1\|_{\mathbb{H}} S_{n,d}(\mathbf{z}) \\
&\le \|k_1\|_{\mathbb{H}} \|(\mathcal{I} - \lambda K)^{-1}\| \|g\|_{\mathbb{H}} S_{n,d}(\mathbf{z}) \\
&\le \mu\beta S_{n,d}(\mathbf{z}),
\end{aligned}
$$

and

$$
\begin{aligned}
II &= \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\} \\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod{n}}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \left|\hat{k}_2(\mathbf{l}) \cdot \hat{u}(\mathbf{h}+\mathbf{l}) 2\pi i (\mathbf{h}+\mathbf{l})\right| \\
&\le \sum_{\mathbf{l}\in\mathbb{Z}^d} \Bigg(\Bigg(\|\vec{\hat{k}}_2(\mathbf{l})\|_\infty^2 \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\} \\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod{n}}} \|\hat{u}(\mathbf{h}+\mathbf{l}) 2\pi i (\mathbf{h}+\mathbf{l})\|_\infty^2 r_\alpha(\gamma, \mathbf{h}+\mathbf{l}) \Bigg)^{1/2} \\
&\qquad \times \Bigg(\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\} \\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod{n}}} \frac{1}{r_\alpha(\gamma, \mathbf{h}+\mathbf{l})} \Bigg)^{1/2} \Bigg)
\end{aligned}
$$

by the Cauchy–Schwarz inequality

$$
\begin{aligned}
&\le \Bigg(\sum_{\mathbf{l}\in\mathbb{Z}^d} \|\vec{\hat{k}}_2(\mathbf{l})\|_\infty^2 \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\} \\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod{n}}} \|\hat{u}(\mathbf{h}+\mathbf{l}) 2\pi i (\mathbf{h}+\mathbf{l})\|_\infty^2 r_\alpha(\gamma, \mathbf{h}+\mathbf{l}) \Bigg)^{1/2} \\
&\qquad \times \Bigg(\sum_{\mathbf{l}\in\mathbb{Z}^d} \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\} \\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod{n}}} \frac{1}{r_\alpha(\gamma, \mathbf{h}+\mathbf{l})} \Bigg)^{1/2} \\
&= \Bigg(\sum_{\mathbf{l}\in\mathbb{Z}^d} \|\vec{\hat{k}}_2(\mathbf{l})\|_\infty^2 \Bigg)^{1/2} \|\nabla u\|_{\mathbb{H},\infty} \Bigg(\sum_{\mathbf{l}\in\mathbb{Z}^d} \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\} \\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod{n}}} \frac{1}{r_\alpha(\gamma, \mathbf{h}+\mathbf{l})} \Bigg)^{1/2} \\
&= \|\nabla u\|_{\mathbb{H},\infty} \Bigg(\sum_{\mathbf{l}\in\mathbb{Z}^d} \|\vec{\hat{k}}_2(\mathbf{l})\|_\infty^2 r_\alpha(\gamma, \mathbf{l}) \Bigg)^{1/2} \\
&\qquad \times \Bigg(\sum_{\mathbf{l}\in\mathbb{Z}^d} \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\} \\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod{n}}} \frac{1}{r_\alpha(\gamma, \mathbf{l}) r_\alpha(\gamma, \mathbf{h}+\mathbf{l})} \Bigg)^{1/2} \\
&= \|\nabla u\|_{\mathbb{H},\infty} \|\vec{k_2}\|_{\mathbb{H}} S_{n,d}(\mathbf{z}) \\
&\le \beta S_{n,d}(\mathbf{z}).
\end{aligned}
$$

In the above equation, we assume that $\|\nabla u\|_{\mathbb{H},\infty} \leq 1$. Thus, we obtain

$$\|(K - K_n)u\|_{\sup} \leq (\mu + 1) S_{n,d}(\mathbf{z}).$$

Similarly, we write

$$\begin{aligned}
(K - K_n)K_n &= KK_n - K_n K_n \\
&= \int_D k_1(\mathbf{x}-\mathbf{y}) \Bigg(\frac{1}{n} \sum_{j=1}^{n} k_1(\mathbf{y}-\mathbf{t}_j) u(\mathbf{t}_j) \\
&\qquad\qquad + \frac{1}{n} \sum_{j=1}^{n} \overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j) \cdot \nabla u(\mathbf{t}_j) \Bigg) dy \\
&\quad + \int_D \overrightarrow{k_2}(x-y) \Bigg(\frac{1}{n} \sum_{j=1}^{n} \nabla k_1(\mathbf{y}-\mathbf{t}_j) u(\mathbf{t}_j) \\
&\qquad\qquad + \frac{1}{n} \sum_{j=1}^{n} \nabla \overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j) \nabla u(\mathbf{t}_j) \Bigg) d\mathbf{y} \\
&\quad - \frac{1}{n} \sum_{i=1}^{n} k_1(\mathbf{x}-\mathbf{t}_i) \Bigg(\frac{1}{n} \sum_{j=1}^{n} k_1(\mathbf{t}_i-\mathbf{t}_j) u(\mathbf{t}_j) \\
&\qquad\qquad + \frac{1}{n} \sum_{j=1}^{n} \overrightarrow{k_2}(\mathbf{t}_i-\mathbf{t}_j) \nabla u(\mathbf{t}_j) \Bigg) \\
&\quad - \frac{1}{n} \sum_{i=1}^{n} \overrightarrow{k_2}(\mathbf{x}-\mathbf{t}_i) \Bigg(\frac{1}{n} \sum_{j=1}^{n} \nabla k_1(\mathbf{t}_i-t_j) u(\mathbf{t}_j) \\
&\qquad\qquad + \frac{1}{n} \sum_{j=1}^{n} \nabla \overrightarrow{k_2}(\mathbf{t}_i-\mathbf{t}_j) \nabla u(\mathbf{t}_j) \Bigg) \\
&= \frac{1}{n} \sum_{j=1}^{n} \Bigg(\int k_1(\mathbf{x}-\mathbf{y}) k_1(\mathbf{y}-\mathbf{t}_j) u(\mathbf{t}_j) d\mathbf{y} \\
&\qquad\qquad - \frac{1}{n} \sum_{i=1}^{n} k_1(\mathbf{x}-\mathbf{t}_i) k_1(\mathbf{t}_i-\mathbf{t}_j) u(\mathbf{t}_j) \Bigg) \\
&\quad + \frac{1}{n} \sum_{j=1}^{n} \Bigg(\int k_1(\mathbf{x}-\mathbf{y}) \overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j) \nabla u(\mathbf{t}_j) d\mathbf{y} \\
&\qquad\qquad - \frac{1}{n} \sum_{i=1}^{n} k_1(\mathbf{x}-\mathbf{t}_i) \overrightarrow{k_2}(\mathbf{t}_i-\mathbf{t}_j) \nabla u(t_j) \Bigg)
\end{aligned}$$

$$+\frac{1}{n}\sum_{j=1}^{n}\bigg(\int \overrightarrow{k_2}(\mathbf{x}-\mathbf{y})\nabla k_1(\mathbf{y}-\mathbf{t}_j)u(\mathbf{t}_j)d\mathbf{y} - \frac{1}{n}\sum_{i=1}^{n}\overrightarrow{k_2}(\mathbf{x}-\mathbf{t}_i)\nabla k_1(\mathbf{t}_i-\mathbf{t}_j)u(\mathbf{t}_j)\bigg)$$

$$+\frac{1}{n}\sum_{j=1}^{n}\bigg(\int \overrightarrow{k_2}(\mathbf{x}-\mathbf{y})\nabla \overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j)\nabla u(\mathbf{t}_j)d\mathbf{y} - \frac{1}{n}\sum_{i=1}^{n}\overrightarrow{k_2}(\mathbf{x}-\mathbf{t}_i)\nabla \overrightarrow{k_2}(\mathbf{t}_i-\mathbf{t}_j)\nabla u(\mathbf{t}_j)\bigg)$$

$$=: I + II + III + IV.$$

Therefore we have

$$\begin{aligned}\|I+II\| &= \sup_{\mathbf{x}\in D}\frac{1}{n}\sum_{j=1}^{n}\bigg|\int_D k_1(\mathbf{x}-\mathbf{y})k_1(\mathbf{y}-\mathbf{t}_j)d\mathbf{y} - \frac{1}{n}\sum_{i=1}^{n}k_1(\mathbf{x}-\mathbf{t}_i)k_1(\mathbf{t}_i-\mathbf{t}_j)\bigg| \\ &\quad + \sup_{\mathbf{x}\in D}\frac{1}{n}\sum_{j=1}^{n}\bigg|\bigg(\int_D k_1(\mathbf{x}-\mathbf{y})\overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j)d\mathbf{y} - \frac{1}{n}\sum_{i=1}^{n}k_1(\mathbf{x}-\mathbf{t}_i)\overrightarrow{k_2}(\mathbf{t}_i-\mathbf{t}_j)\bigg).\nabla u(\mathbf{t}_j)\bigg| \\ &=: \sup_{\mathbf{x}\in D}\frac{1}{n}\sum_{j=1}^{n}|i| + \sup_{\mathbf{x}\in D}\frac{1}{n}\sum_{j=1}^{n}|ii|\end{aligned}$$

and

$$\begin{aligned}\|III+IV\| &= \sup_{\mathbf{x}\in D}\frac{1}{n}\sum_{j=1}^{n}\bigg|\int_D \overrightarrow{k_2}(\mathbf{x}-\mathbf{y}).\nabla k_1(\mathbf{y}-\mathbf{t}_j)d\mathbf{y} - \frac{1}{n}\sum_{i=1}^{n}\overrightarrow{k_2}(\mathbf{x}-\mathbf{t}_i).\nabla k_1(\mathbf{t}_i-\mathbf{t}_j)\bigg| \\ &\quad + \sup_{\mathbf{x}\in D}\frac{1}{n}\sum_{j=1}^{n}\bigg|\bigg(\int_D \overrightarrow{k_2}(\mathbf{x}-\mathbf{y}).\nabla\overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j)d\mathbf{y} - \frac{1}{n}\sum_{i=1}^{n}\overrightarrow{k_2}(\mathbf{x}-\mathbf{t}_i).\nabla\overrightarrow{k_2}(\mathbf{t}_i-\mathbf{t}_j)\bigg).\nabla u(\mathbf{t}_j)\bigg|\end{aligned}$$

$$=: \sup_{\mathbf{x}\in D}\frac{1}{n}\sum_{j=1}^{n}|iii| + \sup_{\mathbf{x}\in D}\frac{1}{n}\sum_{j=1}^{n}|iv|,$$

where

$$i = -\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \int_D k_1(\mathbf{x}-\mathbf{y})k_1(\mathbf{y}-\mathbf{t}_j)e^{-2\pi i\mathbf{h}.\mathbf{y}}d\mathbf{y}$$

$$= -\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \int_D \left(\sum_{\mathbf{l}\in\mathbb{Z}^d}\hat{k}_1(\mathbf{l})e^{2\pi i\mathbf{l}.(\mathbf{x}-\mathbf{y})}\right)k_1(\mathbf{y}-\mathbf{t}_j)e^{-2\pi i\mathbf{h}.\mathbf{y}}d\mathbf{y}$$

$$= -\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d}\hat{k}_1(\mathbf{l})e^{2\pi i\mathbf{l}.\mathbf{x}}\int_D k_1(\mathbf{y}-\mathbf{t}_j)e^{-2\pi i(\mathbf{h}+\mathbf{l}).\mathbf{y}}d\mathbf{y}$$

$$= -\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d}\hat{k}_1(\mathbf{l})e^{2\pi i\mathbf{l}.\mathbf{x}}\hat{k}_1(\mathbf{h}+\mathbf{l})e^{-2\pi i(\mathbf{h}+\mathbf{l}).\mathbf{t}_j},$$

$$ii = -\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \int_D k_1(\mathbf{x}-\mathbf{y})\overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j).\nabla u(\mathbf{t}_j)e^{-2\pi i\mathbf{h}.\mathbf{y}}d\mathbf{y}$$

$$= -\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d}\hat{k}_1(\mathbf{l})e^{2\pi i\mathbf{l}.\mathbf{x}}\int_D \overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j).\nabla u(\mathbf{t}_j)e^{-2\pi i(\mathbf{h}+\mathbf{l}).\mathbf{y}}d\mathbf{y}$$

$$= -\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d}\hat{k}_1(\mathbf{l})e^{2\pi i\mathbf{l}.\mathbf{x}}\hat{k}_3(\mathbf{h}+\mathbf{l})e^{-2\pi i(\mathbf{h}+\mathbf{l}).\mathbf{t}_j},$$

where we define $k_3(\mathbf{y}-\mathbf{t}_j) = \overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j)\cdot\nabla u(\mathbf{t}_j)$,

$$iii = -\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \int_D \overrightarrow{k_2}(\mathbf{x}-\mathbf{y}).\nabla k_1(\mathbf{y}-\mathbf{t}_j)e^{-2\pi i\mathbf{h}.\mathbf{y}}d\mathbf{y}$$

$$= -\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \int_D \left(\sum_{\mathbf{l}\in\mathbb{Z}^d}\hat{\overrightarrow{k_2}}(\mathbf{l})e^{2\pi i\mathbf{l}.(\mathbf{x}-\mathbf{y})}\right).\nabla k_1(\mathbf{y}-\mathbf{t}_j)e^{-2\pi i\mathbf{h}.\mathbf{y}}d\mathbf{y}$$

$$= -\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d}\hat{\overrightarrow{k_2}}(\mathbf{l})e^{2\pi i\mathbf{l}.\mathbf{x}}.\int_D \nabla k_1(\mathbf{y}-\mathbf{t}_j)e^{-2\pi i(\mathbf{h}+\mathbf{l}).\mathbf{y}}d\mathbf{y}$$

$$= - \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \overrightarrow{\hat{k}_2}(\mathbf{l}) e^{2\pi i \mathbf{l}.\mathbf{x}} . \nabla \hat{k}_1(\mathbf{h}+\mathbf{l}) e^{-2\pi i (\mathbf{h}+\mathbf{l}).\mathbf{t}_j},$$

$$iv = - \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \int_D \overrightarrow{k_2}(\mathbf{x}-\mathbf{y}).\nabla\overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j).\nabla u(\mathbf{t}_j) e^{-2\pi i \mathbf{h}.\mathbf{y}} d\mathbf{y}$$

$$= - \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \int_D \left(\sum_{\mathbf{l}\in\mathbb{Z}^d} \overrightarrow{\hat{k}_2}(\mathbf{l}) e^{2\pi i \mathbf{l}.(\mathbf{x}-\mathbf{y})} \right) . \nabla\overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j).\nabla u(\mathbf{t}_j) e^{-2\pi i \mathbf{h}.\mathbf{y}} d\mathbf{y}$$

$$= - \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \overrightarrow{\hat{k}_2}(\mathbf{l}) e^{2\pi i \mathbf{l}.\mathbf{x}} . \int_D \nabla\overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j).\nabla u(\mathbf{t}_j) e^{-2\pi i (\mathbf{h}+\mathbf{l}).\mathbf{y}} d\mathbf{y}$$

$$= - \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \overrightarrow{\hat{k}_2}(\mathbf{l}) e^{2\pi i \mathbf{l}.\mathbf{x}} . \overrightarrow{\hat{k}_4}(\mathbf{h}+\mathbf{l}) e^{-2\pi i (\mathbf{h}+\mathbf{l}).\mathbf{t}_j},$$

where $\overrightarrow{k_4} = \nabla\overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j)\nabla u(\mathbf{t}_j)$. Therefore we write

$$\|(K-K_n)K_n\|$$

$$\le \frac{1}{n} \sup_{\mathbf{x}\in D} \sum_{j=1}^{n} \Bigg| \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \hat{k}_1(\mathbf{l})\hat{k}_1(\mathbf{h}+\mathbf{l}) e^{2\pi i \mathbf{l}.\mathbf{x}} e^{-2\pi i (\mathbf{h}+\mathbf{l}).\mathbf{t}_j} \Bigg|$$

$$+ \frac{1}{n} \sup_{\mathbf{x}\in D} \sum_{j=1}^{n} \Bigg| \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{l\in\mathbb{Z}^d} \hat{k}_1(l)\hat{k}_3(\mathbf{h}+\mathbf{l}) e^{2\pi i \mathbf{l}.\mathbf{x}} e^{-2\pi i (\mathbf{h}+\mathbf{l}).\mathbf{t}_j} \Bigg|$$

$$+ \frac{1}{n} \sup_{\mathbf{x}\in D} \sum_{j=1}^{n} \Bigg| \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \overrightarrow{\hat{k}_2}(\mathbf{l}).\nabla\hat{k}_1(\mathbf{h}+\mathbf{l}) e^{2\pi i \mathbf{l}.\mathbf{x}} e^{-2\pi i (\mathbf{h}+\mathbf{l}).\mathbf{t}_j} \Bigg|$$

$$+ \frac{1}{n} \sup_{\mathbf{x}\in D} \sum_{j=1}^{n} \Bigg| \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \overrightarrow{\hat{k}_2}(\mathbf{l}).\overrightarrow{\hat{k}_4}(\mathbf{h}+\mathbf{l}) e^{2\pi i \mathbf{l}.\mathbf{x}} e^{-2\pi i (\mathbf{h}+\mathbf{l}).\mathbf{t}_j} \Bigg|$$

$$\le \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \Big(|\hat{k}_1(\mathbf{l})||\hat{k}_1(\mathbf{h}+\mathbf{l})| + |\hat{k}_1(\mathbf{l})||\overrightarrow{\hat{k}_4}(\mathbf{h}+\mathbf{l})| + |\overrightarrow{\hat{k}_2}(\mathbf{l})\cdot\nabla\hat{k}_1(\mathbf{h}+\mathbf{l})| + |\overrightarrow{\hat{k}_2}(\mathbf{l})\cdot\overrightarrow{k_4}(\mathbf{h}+\mathbf{l})| \Big)$$

$$\leq \Big(\|k_1\|_{\mathbb{H}}^2 + \|k_1\|_{\mathbb{H}}\|k_3\| + \|\overrightarrow{k_2}\|_{\mathbb{H},\infty}\|\nabla k_1\|_{\mathbb{H},\infty} + |\overrightarrow{k_2}\|_{\mathbb{H},\infty}\|\overrightarrow{k_4}\|_{\mathbb{H},\infty}\Big) S_{n,d}(\mathbf{z})$$

$$\leq \Big(\|k_1\|_{\mathbb{H}}^2 + \|k_1\|_{\mathbb{H}}\|\overrightarrow{k_2}\|_{\mathbb{H}} \cdot \|\nabla u\|_{\mathbb{H},\infty} + \|\overrightarrow{k_2}\|_{\mathbb{H},\infty}\|\nabla k_1\|_{\mathbb{H},\infty} + \|\overrightarrow{k_2}\|_{\mathbb{H},\infty}\|\nabla\overrightarrow{k_2}\|_{\mathbb{H},\infty}\|\nabla u\|_{\mathbb{H},\infty}\Big) S_{n,d}(\mathbf{z}).$$

If $k_1, \overrightarrow{k_2} \in Æ$ and $\|\nabla u\|_{\mathbb{H},\infty} \leq 1$, then we have

$$\|(K - K_n)K_n\| \leq (\beta^2 + \beta^2 + \beta^2 + \beta^2) S_{n,d}(\mathbf{z}) = 4\beta^2 S_{n,d}(\mathbf{z}).$$

On the other hand, we write

$$\Delta_n = \|(\mathcal{I} - \lambda K)^{-1}\|\|(K - K_n)K_n\| \leq 4\mu\beta^2 S_{n,d}(\mathbf{z}),$$

so we have $\Delta_n < 1$. Hence we write

$$S_{n,d}(\mathbf{z}) < \frac{1}{4\mu\beta^2}.$$

Also, we observe that

$$\begin{aligned}
\|(\mathcal{I} - \lambda K_n)^{-1}\| &\leq \frac{1 + \|(\mathcal{I} - \lambda K_n)^{-1}\|(\|k_1\|_{\mathbb{H}} + \|k_3\|_{\mathbb{H}}) \prod_{j=1}^d \big(1 + 2\zeta(\alpha)\gamma_j\big)^{1/2}}{1 - \Delta_n} \\
&\leq \frac{1 + 2\mu\beta \prod_{j=1}^d \big(1 + 2\zeta(\alpha)\gamma_j\big)^{1/2}}{1 - 4\mu\beta^2 S_{n,d}(\mathbf{z})} \\
&\leq \frac{1 + 2\mu\beta}{1 - 4\mu\beta^2 S_{n,d}(\mathbf{z})} \prod_{j=1}^d \big(1 + 2\zeta(\alpha)\gamma_j\big)^{1/2}. \qquad \text{(B.1)}
\end{aligned}$$

On the other hand, we conclude that

$$\begin{aligned}
\|u - u_n\|_{\sup} &\leq \lambda\|(\mathcal{I} - \lambda K_n)^{-1}\|\|(K - K_n)u\|_{\sup} \\
&\leq \frac{\lambda(1 + 2\mu\beta)}{1 - 4\mu\beta^2 S_{n,d}(\mathbf{z})} \prod_{j=1}^d \big(1 + 2\zeta(\alpha)\gamma_j\big)^{1/2} \|(K - K_n)u\|_{\sup} \\
&\leq \frac{\lambda(1 + 2\mu\beta)}{1 - 4\mu\beta^2 S_{n,d}(\mathbf{z})} \prod_{j=1}^d \big(1 + 2\zeta(\alpha)\gamma_j\big)^{1/2} \\
&\qquad \times \big[\|(\mathcal{I} - \lambda K)^{-1}\|\|g\|_{\mathbb{H}}\|k_1\|_{\mathbb{H}} + \beta\big] S_{n,d}(\mathbf{z}) \\
&\leq \frac{\lambda(1 + 2\mu\beta) \prod_{j=1}^d \big(1 + 2\zeta(\alpha)\gamma_j\big)^{1/2}}{1 - 4\mu\beta^2 S_{n,d}(\mathbf{z})} (\mu\beta + \beta) S_{n,d}(\mathbf{z}).
\end{aligned}$$

C Appendix 3. Proof of Proposition 3.1

Again we consider $k_1(\mathbf{x}-\mathbf{y}) = k_1$ as a constant function such that

$$\|k_1\|_{\mathbb{H}} = k_1 \le \beta \quad \text{and} \quad \|\widetilde{K}^{-1}\| \le \mu,$$

also, we assume that $k_1(x-y)$ and $\overrightarrow{k_2}(\mathbf{x}-\mathbf{y})$ are separable kernels. Then, we have

$$\overrightarrow{a_1} \cdot \nabla u(\mathbf{x}) = g(\mathbf{x}) + \lambda_1 \int_D k_1(\mathbf{x}-\mathbf{y}) u(\mathbf{y}) d\mathbf{y} + \lambda_2 \int_D \overrightarrow{k_2}(\mathbf{x}-\mathbf{y}) \cdot \nabla u(\mathbf{y}) d\mathbf{y},$$

or we write

$$\begin{aligned}\overrightarrow{a_1} \cdot \nabla u(\mathbf{x}) &= g(\mathbf{x}) + \lambda_1 \alpha_1(\mathbf{x}) \int_D b_1(\mathbf{y}) u(\mathbf{y}) d\mathbf{y} + \lambda_2 \alpha_2(\mathbf{x}) \int_D b_2(\mathbf{y}) \cdot \nabla u(\mathbf{y}) d\mathbf{y}, \\ &= g(\mathbf{x}) + \lambda_1 c_1 + \lambda_2 c_2.\end{aligned}$$

We find c_1 and c_2 by integration of the above equation, thus we have

$$u(\mathbf{x}) = \int g(\mathbf{x}) d\mathbf{x} + (\lambda_1 c_1 + \lambda_2 c_2)\mathbf{x} + d_1.$$

Hence we have

$$\begin{aligned}g(\mathbf{x}) + \lambda_1 c_1 + \lambda_2 c_2 = g(\mathbf{x}) + \lambda_1 \int_D k_1 \Big(\int g(\mathbf{y}) d\mathbf{y} + (\lambda_1 c_1 + \lambda_2 c_2)\mathbf{y} + d_1 \Big) d\mathbf{y} \\ + \lambda_2 \int_D \big(g(\mathbf{y}) + \lambda_1 c_1 + \lambda_2 c_2 \big) d\mathbf{y},\end{aligned}$$

therefore

$$\begin{aligned}c_2 &= \frac{1}{\lambda_2 - \lambda_1 \lambda_2 \frac{K_1}{2} - \lambda_2^2} \Bigg(\lambda_1 k_1 \Big[\int_D \int g(\mathbf{y}) d\mathbf{y} d\mathbf{y} + \Big(\frac{\lambda_1 c_1}{2} + d_1 \Big) \Big] \\ &\qquad + \lambda_2 \Big(\int_D g(\mathbf{y}) d\mathbf{y} + \lambda_1 c_1 \Big) + d_2 - c_1 - \lambda_1 c_1 \Bigg) \\ &= l(c_1).\end{aligned}$$

Then we have

$$\begin{aligned}u(\mathbf{y}) &= \int g(\mathbf{y}) d\mathbf{y} + \lambda_1 c_1 \mathbf{y} \\ &\quad + \frac{1}{\lambda_2 - \lambda_1 \lambda_2 \frac{K_1}{2} - \lambda_2^2} \Bigg(\lambda_1 k_1 \mathbf{y} \Big[\int_D \int g(\mathbf{y}) d\mathbf{y} d\mathbf{y} + \Big(\frac{\lambda_1 c_1}{2} + d_1 \Big) \Big] \\ &\qquad + \lambda_2 \Big(\int_D g(\mathbf{y}) d\mathbf{y} + \lambda_1 c_1 \Big) + d_2 - c_1 - \lambda_1 c_1 \Bigg),\end{aligned}$$

and

$$u_n(\mathbf{y}) = \int g(\mathbf{y})d\mathbf{y} + \lambda_1 c_1 \mathbf{y} + \frac{1}{\lambda_2 - \lambda_1\lambda_2\frac{K_1}{2} - \lambda_2^2}\Bigg(\lambda_1 k_1 \mathbf{y}\Bigg[\frac{1}{n}\sum_{i=1}^{n}\Big(\int g(\mathbf{y})d\mathbf{y}\Big)_{\mathbf{t}_i} + \Big(\frac{\lambda_1 c_1}{2} + d_1\Big)\Bigg] + \frac{\lambda_2}{n}\sum_{i=1}^{n} g(\mathbf{t}_i) + \lambda_2\lambda_1 c_1 + d_2 - c_1 - \lambda_1 c_1\Bigg)$$

and so

$$u - u_n = \frac{1}{\lambda_2 - \lambda_1\lambda_2\frac{K_1}{2} - \lambda_2^2}\Bigg(\lambda_1\lambda_2 k_1 \mathbf{y}\Bigg[\int_D \int g(\mathbf{y})d\mathbf{y}d\mathbf{y} - \frac{1}{n}\sum_{i=1}^{n}\Big(\int g(\mathbf{y})d\mathbf{y}\Big)_{\mathbf{t}_i}\Bigg] + \lambda_2\Big(\int_D g(\mathbf{y})d\mathbf{y} - \frac{1}{n}\sum_{i=1}^{n} g(\mathbf{t}_i)\Big)\Bigg).$$

Hence we have

$$\begin{aligned} e_{n,d}(A_n) &\geq \sup_{\|g\|_{\mathbb{H}}\leq 1} \|S(g,k_1,k_2) - A_n(g,k_1,k_2)\|_{\sup} \\ &= \frac{\lambda_1 k_1}{1 - \frac{k_1\lambda_1}{2} - \lambda}\Bigg(\int_D \int g(\mathbf{y})d\mathbf{y}d\mathbf{y} - \frac{1}{n}\sum_{i=1}^{n}\Big(\int g(\mathbf{y})d\mathbf{y}\Big)_{\mathbf{t}_i} + e_{n,d}^{\text{wor-int}}(\mathbf{t}_1,\dots,\mathbf{t}_n)\Bigg), \end{aligned}$$

so we have

$$e_{n,d}(A_n) \geq \frac{\lambda_1 k_1}{1 - \frac{k_1\lambda_1}{2} - \lambda_1}\Bigg(\overbrace{\int_D \int g(\mathbf{y})d\mathbf{y}d\mathbf{y} - \frac{1}{n}\sum_{i=1}^{n}\Big(\int g(\mathbf{y})d\mathbf{y}\Big)_{\mathbf{t}_i}}^{=\star} + e_{n,d}^{\text{wor-int}}(\mathbf{t}_1,\dots,\mathbf{t}_n)\Bigg).$$

On the other hand, by using (2.10) and (2.11), we have

$$e_{n,d}(A_n) \geq \frac{\lambda_1 k_1}{1 - \frac{k_1 \lambda_1}{2} - \lambda_1} \max\Bigg(\bigstar + \left(\frac{2\delta(\alpha)\gamma_1}{n^\alpha}\right)^{\frac{1}{2}}, \\ \bigstar + \left(\frac{1}{n}\prod_{j=1}^{d}(1 + 2\delta(\alpha) w_\alpha \gamma_j) - 1\right)^{\frac{1}{2}}\Bigg).$$

We know that if $\widetilde{K}_n u_n = \overrightarrow{a_1} \cdot \nabla u - g$, then we have

$$u_n = \widetilde{K}_n^{-1}(\overrightarrow{a_1} \cdot \nabla u - g).$$

Also, we have $\widetilde{K_n} u = \overrightarrow{a_1} \cdot \nabla u - g$ and then

$$u = \widetilde{K_n}^{-1}(\overrightarrow{a_1} \cdot \nabla u - g).$$

Moreover, if we write $\widetilde{K}_n u = (\widetilde{K}_n - \widetilde{K})u + \widetilde{K}u$ or

$$u = \widetilde{K}_n^{-1}(\widetilde{K}_n - \widetilde{K})u + \widetilde{K}_n^{-1}\widetilde{K}u,$$

then we have

$$u_n - u = \widetilde{K}_n^{-1}\widetilde{K}u - \widetilde{K}_n^{-1}(\widetilde{K}_n - \widetilde{K})u - \widetilde{K}_n^{-1}\widetilde{K}u = -\widetilde{K}_n^{-1}(\widetilde{K}_n - \widetilde{K})u,$$

or

$$u_n - u = \widetilde{K}_n^{-1}(\widetilde{K} - \widetilde{K}_n)u. \tag{C.1}$$

Therefore we conclude the following inequality:

$$\|S(g, k_1, \overrightarrow{k_2}) - A_n(g, k_1, \overrightarrow{k_2})\| = \|u - u_n\| \leq \|\widetilde{K_n}^{-1}\| \|(\widetilde{K} - \widetilde{K}_n)u\|_{\sup}. \tag{C.2}$$

We recall that

$$\Delta_n := \|\widetilde{K}^{-1}\| \|(\widetilde{K} - \widetilde{K}_n)\widetilde{K}_n\| < 1,$$

therefore we write

$$\|\widetilde{K}_n^{-1}\| \leq \frac{1 + \|\widetilde{K}^{-1}\| \|\widetilde{K}_n\|}{1 - \Delta_n}. \tag{C.3}$$

On the other hand, we have

$$\|\widetilde{K}_n\| \leq \lambda_1 \|k_1\|_{\sup} + \lambda_2 \|k_3\|_{\sup} \\ \leq (\lambda_1 \|k_1\|_{\mathbb{H}} + \lambda_2 \|k_3\|_{\mathbb{H}}) \prod_{j=1}^{d}(1 + 2\zeta(\alpha)\gamma_j). \tag{C.4}$$

Hence we write the following inequality:

$$\begin{aligned}
\|u-u_n\|_{\sup} &\le \|\widetilde{K}_n^{-1}\| \|(\widetilde{K}-\widetilde{K}_n)u\|_{\sup} \\
&\le \frac{1+\|\widetilde{K}^{-1}\|\cdot\|\widetilde{K}_n\|}{1-\Delta_n}\|(\widetilde{K}-\widetilde{K}_n)u\|_{\sup} \\
&\le \frac{1+\|\widetilde{K}^{-1}\|(\lambda_1\|k_1\|_{\mathbb{H}}+\lambda_2\|k_3\|_{\mathbb{H}})\prod_{j=1}^{d}\big(1+2\zeta(\alpha)\gamma_j\big)^{1/2}}{1-\|\widetilde{K}^{-1}\|\cdot\|(\widetilde{K}-\widetilde{K}_n)\widetilde{K}_n\|} \\
&\qquad \times \|(\widetilde{K}-\widetilde{K}_n)u\|_{\sup}.
\end{aligned}$$

Moreover, we know that

$$\begin{aligned}
(\widetilde{K}-\widetilde{K}_n)u(\mathbf{x}) &= \lambda_1\int_D k_1(\mathbf{x},\mathbf{y})u(\mathbf{y})d\mathbf{y}+\lambda_2\int_D \overrightarrow{k_2}(\mathbf{x},\mathbf{y})\cdot\nabla u(\mathbf{y})d\mathbf{y} \\
&\quad -\frac{\lambda_1}{n}\sum_{i=1}^{n}k_1(\mathbf{x}-\mathbf{t}_i)u(t_i)+\frac{\lambda_2}{n}\sum_{i=1}^{n}\overrightarrow{k_2}(\mathbf{x}-\mathbf{t}_i)\cdot\nabla u(\mathbf{t}_i) \\
&= -\lambda_1\sum_{\substack{h\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}}\hat{U}_{1\mathbf{x}}(\mathbf{h})-\lambda_2\sum_{\substack{h\in\mathbb{Z}^d-\{0\}\\ h\cdot z\equiv 0 \pmod n}}\hat{U}_{2\mathbf{x}}(\mathbf{h})
\end{aligned}$$

where $U_{1\mathbf{x}}=k_1(\mathbf{x}-\mathbf{y})u(\mathbf{y})$, $U_{2\mathbf{x}}=\overrightarrow{k_2}(\mathbf{x}-\mathbf{y}).\nabla u(\mathbf{y})$,

$$\begin{aligned}
\hat{U}_{1\mathbf{x}} &= \int_D k_1(\mathbf{x}-\mathbf{y})u(\mathbf{y})e^{-2\pi i\mathbf{h}.\mathbf{y}}d\mathbf{y} \\
&= \int_D\Big(\sum_{\mathbf{l}\in\mathbb{Z}^d}\hat{k}_1(\mathbf{l})e^{2\pi i\mathbf{l}.(\mathbf{x}-\mathbf{y})}\Big)u(\mathbf{y})e^{-2\pi i\mathbf{h}.\mathbf{y}}d\mathbf{y} \\
&= \sum_{\mathbf{l}\in\mathbb{Z}^d}\hat{k}_1(\mathbf{l})e^{2\pi i\mathbf{l}.\mathbf{x}}\int_D u(\mathbf{y})e^{-2\pi i(\mathbf{h}+\mathbf{l}).\mathbf{y}}d\mathbf{y} \\
&= \sum_{\mathbf{l}\in\mathbb{Z}^d}\hat{k}_1(\mathbf{l})\hat{u}(\mathbf{h}+\mathbf{l})e^{2\pi i\mathbf{l}.\mathbf{x}},
\end{aligned}$$

and

$$\begin{aligned}
\hat{U}_{2\mathbf{x}} &= \int_D \overrightarrow{k_2}(\mathbf{x}-\mathbf{y})\cdot\nabla u(\mathbf{y})e^{-2\pi i\mathbf{h}.\mathbf{y}}d\mathbf{y} \\
&= \int_D\Big(\sum_{\mathbf{l}\in\mathbb{Z}^d}\overrightarrow{\hat{k}_2}(\mathbf{l})e^{2\pi i\mathbf{l}.(\mathbf{x}-\mathbf{y})}\Big)\cdot\nabla u(\mathbf{y})e^{-2\pi i\mathbf{h}.\mathbf{y}}d\mathbf{y}
\end{aligned}$$

$$= \sum_{\mathbf{l}\in\mathbb{Z}^d} \overrightarrow{\hat{k}_2}(\mathbf{l}) e^{2\pi i \mathbf{l}.\mathbf{x}} \int_G \nabla u(\mathbf{y}) e^{-2\pi i (\mathbf{h}+\mathbf{l})\mathbf{y}} d\,\mathbf{y}$$

$$= \sum_{\mathbf{l}\in\mathbb{Z}^d} \overrightarrow{\hat{k}_2}(\mathbf{l}) e^{2\pi i \mathbf{l}.\mathbf{x}} \hat{u}(\mathbf{h}+\mathbf{l}) 2\pi i (\mathbf{h}+\mathbf{l}).$$

Thus, we have

$$\begin{aligned}
\|(\widetilde{K}-\widetilde{K}_n)u\|_{\sup} &= \sup_{\mathbf{x}\in D}\Bigg| \lambda_1 \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \hat{U}_{1\mathbf{x}}(\mathbf{h}) + \lambda_2 \sum_{\substack{h\in\mathbb{Z}^d-\{0\}\\ h\cdot z\equiv 0 \pmod n}} \hat{U}_{2\mathbf{x}}(\mathbf{h}) \Bigg| \\
&= \sup_{\mathbf{x}\in D}\Bigg| \lambda_1 \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \hat{k}_1(\mathbf{l}) \hat{u}(\mathbf{h}+\mathbf{l}) e^{2\pi i \mathbf{l}.\mathbf{x}} \\
&\qquad + \lambda_2 \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \overrightarrow{\hat{k}_2}(\mathbf{l}) \hat{u}(\mathbf{h}+\mathbf{l}) e^{2\pi i \mathbf{l}.\mathbf{x}} 2\pi i (\mathbf{h}+\mathbf{l}) \Bigg| \\
&\le \overbrace{\lambda_1 \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} |\hat{k}_1(\mathbf{l})| |\hat{u}(\mathbf{h}+\mathbf{l})|}^{=I_0} \\
&\qquad + \underbrace{\lambda_2 \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} |\overrightarrow{\hat{k}_2}(\mathbf{l}) \hat{u}(\mathbf{h}+\mathbf{l}) 2\pi i (\mathbf{h}+\mathbf{l})|}_{=II_0}.
\end{aligned}$$

Therefore we compute

$$\begin{aligned}
I_0 &\le \lambda_1 \sum_{\mathbf{l}\in\mathbb{Z}^d} \Bigg[|\hat{k}_1(\mathbf{l})| \Bigg(\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} |\hat{u}(\mathbf{h}+\mathbf{l})|^2 r_\alpha(\gamma, \mathbf{h}+\mathbf{l}) \Bigg)^{1/2} \\
&\qquad \times \Bigg(\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \frac{1}{r_\alpha(\gamma, \mathbf{h}+\mathbf{l})} \Bigg)^{1/2} \Bigg] \\
&\le \lambda_1 \Bigg(\sum_{\mathbf{l}\in\mathbb{Z}^d} |\hat{k}_1(\mathbf{l})|^2 \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} |\hat{u}(\mathbf{h}+\mathbf{l})|^2 r_\alpha(\gamma, \mathbf{h}+\mathbf{l}) \Bigg)^{1/2}
\end{aligned}$$

$$\times \left(\sum_{\mathbf{l}\in\mathbb{Z}^d} \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\} \\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \frac{1}{r_\alpha(\gamma, \mathbf{h}+\mathbf{l})} \right)^{1/2}$$

$$= \lambda_1 \left(\sum_{\mathbf{l}\in\mathbb{Z}^d} |\hat{k}_1(\mathbf{l})| \|u\|_{\mathbb{H}} \right) \left(\sum_{\mathbf{l}\in\mathbb{Z}^d} \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\} \\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \frac{1}{r_\alpha(\gamma, \mathbf{h}+\mathbf{l})} \right)$$

$$= \lambda_1 \|u\|_{\mathbb{H}} \left(\sum_{\mathbf{l}\in\mathbb{Z}^d} |\hat{k}_1(\mathbf{l})|^2 r_\alpha(\gamma, \mathbf{l}) \right)^{1/2}$$

$$\times \left(\sum_{\mathbf{l}\in\mathbb{Z}^d} \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\} \\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \frac{1}{r_\alpha(\gamma, \mathbf{l}) \cdot r_\alpha(\gamma, \mathbf{h}+\mathbf{l})} \right)^{1/2}$$

$$= \lambda_1 \|u\|_{\mathbb{H}} \|k_1\|_{\mathbb{H}} S_{n,d}(\mathbf{z})$$

$$\leq \lambda_1 \|\widetilde{K}^{-1}\| \|\overrightarrow{a_1} \cdot \nabla u - g\|_{\mathbb{H}} \|k_1\|_{\mathbb{H}} S_{n,d}(\mathbf{z})$$

$$\leq \lambda_1 \mu \beta S_{n,d}(\mathbf{z}).$$

Also, we have

$$II_0 \leq \lambda_2 \sum_{\substack{h\in\mathbb{Z}^d-\{0\} \\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \|\overrightarrow{\hat{k}_2}(\mathbf{l})\|_\infty \|\hat{u}(\mathbf{h}+\mathbf{l}) 2\pi i (\mathbf{h}+\mathbf{l})\|_\infty$$

$$\leq \lambda_2 \sum_{\mathbf{l}\in\mathbb{Z}^d} \Bigg[\|\overrightarrow{\hat{k}_2}(\mathbf{l})\|_\infty \Bigg(\sum_{\substack{h\in\mathbb{Z}^d-\{0\} \\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \|\hat{u}(\mathbf{h}+\mathbf{l}) 2\pi i (\mathbf{h}+\mathbf{l})\|_\infty^2 r_\alpha(\gamma, \mathbf{h}+\mathbf{l}) \Bigg)^{1/2}$$

$$\times \Bigg(\sum_{\substack{h\in\mathbb{Z}^d-\{0\} \\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \frac{1}{r_\alpha(\gamma, \mathbf{h}+\mathbf{l})} \Bigg)^{1/2} \Bigg]$$

$$\leq \lambda_2 \|\nabla u\|_{\mathbb{H},\infty} \left(\sum_{\mathbf{l}\in\mathbb{Z}^d} \|\overrightarrow{\hat{k}_2}(\mathbf{l})\|_\infty^2 r_\alpha(\gamma, \mathbf{h}) \right)^{1/2}$$

$$\times \left(\sum_{\mathbf{l}\in\mathbb{Z}^d} \sum_{\substack{h\in\mathbb{Z}^d-\{0\} \\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \frac{1}{r_\alpha(\gamma, \mathbf{h}) r_\alpha(\gamma, \mathbf{h}+\mathbf{l})} \right)^{1/2}$$

$$\leq \lambda_2 \|\nabla u\|_{\mathbb{H},\infty} \|\overrightarrow{k_2}\|_{\mathbb{H},\infty} S_{n,d}(\mathbf{z})$$

$$\leq \lambda_2 \beta \|\nabla u\|_{\mathbb{H},\infty} S_{n,d}(\mathbf{z}). \tag{C.5}$$

In (C.5), we assume $\|\nabla u\|_{\mathbb{H},\infty} \le 1$. Therefore we have

$$\|(\widetilde{K} - \widetilde{K}_n)u\|_{\sup} \le (\lambda_1\mu + \lambda_2)\beta S_{n,d}(\mathbf{z}). \tag{C.6}$$

Using a similar argument, we obtain

$$\begin{aligned}
(\widetilde{K} - \widetilde{K}_n)\widetilde{K}_n u &= \widetilde{K}\widetilde{K}_n u - \widetilde{K}_n\widetilde{K}_n u \\
&= \lambda_1 \int_G k_1(\mathbf{x}-\mathbf{y})\Bigg(\frac{\lambda_1}{n}\sum_{j=1}^n k_1(\mathbf{y}-\mathbf{t}_j)u(\mathbf{t}_j) \\
&\qquad + \frac{\lambda_2}{n}\sum_{j=1}^n \overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j)\nabla u(\mathbf{t}_j)\Bigg)d\mathbf{y} \\
&\quad + \lambda_2 \int_D \overrightarrow{k_2}(\mathbf{x}-\mathbf{y})\Bigg(\frac{\lambda_1}{n}\sum_{j=1}^n \nabla k_1(\mathbf{y}-\mathbf{t}_j)u(\mathbf{t}_j) \\
&\qquad + \frac{\lambda_2}{n}\sum_{j=1}^n \nabla\overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j)\nabla u(\mathbf{t}_j)\Bigg)d\mathbf{y} \\
&\quad - \frac{\lambda_1}{n}\sum_{i=1}^n k_1(\mathbf{x}-\mathbf{t}_i)\Bigg(\frac{\lambda_1}{n}\sum_{j=1}^n k_1(\mathbf{t}_i-\mathbf{t}_j)u(\mathbf{t}_j) \\
&\qquad + \frac{\lambda_2}{n}\sum_{j=1}^n \overrightarrow{k_2}(\mathbf{t}_i-\mathbf{t}_j)\nabla u(t_j)\Bigg) \\
&\quad - \frac{\lambda_2}{n}\sum_{i=1}^n \overrightarrow{k_2}(\mathbf{x}-\mathbf{t}_i)\Bigg(\frac{\lambda_1}{n}\sum_{j=1}^n \nabla k_1(t_i-t_j)u(t_j) \\
&\qquad + \frac{\lambda_2}{n}\sum_{j=1}^n \nabla\overrightarrow{k_2}(\mathbf{t}_i-\mathbf{t}_j)\nabla u(\mathbf{t}_j)\Bigg) \\
&= \frac{\lambda_1}{n}\sum_{j=1}^n\Bigg(\lambda_1\int_D k_1(\mathbf{x}-\mathbf{y})k_1(\mathbf{y}-\mathbf{t}_j)u(\mathbf{t}_j)d\mathbf{y} \\
&\qquad - \frac{\lambda_1}{n}\sum_{i=1}^n k_1(\mathbf{x}-\mathbf{t}_i)k_1(\mathbf{t}_i-\mathbf{t}_j)u(\mathbf{t}_j)\Bigg) \\
&\quad + \frac{\lambda_2}{n}\sum_{j=1}^n\Bigg(\lambda_1\int_D k_1(\mathbf{x}-\mathbf{y})\overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j)\nabla u(\mathbf{t}_j)d\mathbf{y} \\
&\qquad - \frac{\lambda_1}{n}\sum_{i=1}^n k_1(\mathbf{x}-\mathbf{t}_i)\overrightarrow{k_2}(\mathbf{t}_i-\mathbf{t}_j)\nabla u(\mathbf{t}_j)\Bigg)
\end{aligned}$$

$$+\frac{\lambda_1}{n}\sum_{j=1}^{n}\Bigg(\lambda_2\int_G \overrightarrow{k_2}(\mathbf{x}-\mathbf{y})\nabla k_1(\mathbf{y}-\mathbf{t}_j)u(\mathbf{t}_j)d\mathbf{y}$$

$$-\frac{\lambda_2}{n}\sum_{i=1}^{n}\overrightarrow{k_2}(\mathbf{x}-\mathbf{t}_i)\nabla k_1(\mathbf{t}_i-\mathbf{t}_j)u(\mathbf{t}_j)\Bigg)$$

$$+\frac{\lambda_2}{n}\sum_{j=1}^{n}\Bigg(\lambda_2\int_G \overrightarrow{k_2}(\mathbf{x}-\mathbf{y})\nabla\overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j)\nabla u(\mathbf{t}_j)d\mathbf{y}$$

$$-\frac{\lambda_2}{n}\sum_{i=1}^{n}\overrightarrow{k_2}(\mathbf{x}-\mathbf{t}_i)\nabla\overrightarrow{k_2}(\mathbf{t}_i-\mathbf{t}_j)\nabla u(\mathbf{t}_j)\Bigg),$$

$$=: I+II+III+IV.$$

We write

$$I+II=\frac{\lambda_1}{n}\sum_{j=1}^{n}\Bigg(\lambda_1\int_D k_1(\mathbf{x}-\mathbf{y})k_1(\mathbf{y}-\mathbf{t}_j)d\mathbf{y}$$

$$-\frac{\lambda_1}{n}\sum_{i=1}^{n}k_1(\mathbf{x}-\mathbf{t}_i)k_1(\mathbf{t}_i-\mathbf{t}_j)\Bigg)u(\mathbf{t}_j)$$

$$+\frac{\lambda_2}{n}\sum_{j=1}^{n}\Bigg(\lambda_1\int_G k_1(\mathbf{x}-\mathbf{y})\overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j)d\mathbf{y}$$

$$-\frac{\lambda_1}{n}\sum_{i=1}^{n}k_1(\mathbf{x}-\mathbf{t}_i)\overrightarrow{k_2}(\mathbf{t}_i-\mathbf{t}_j)\Bigg).\nabla u(\mathbf{t}_j),$$

$$\|I+II\|=\sup_{\mathbf{x}\in D}\frac{\lambda_1}{n}\sum_{j=1}^{n}\Bigg|\lambda_1\int_D k_1(\mathbf{x}-\mathbf{y})k_1(\mathbf{y}-\mathbf{t}_j)d\mathbf{y}$$

$$-\frac{\lambda_1}{n}\sum_{i=1}^{n}k_1(\mathbf{x}-\mathbf{t}_i)k_1(\mathbf{t}_i-\mathbf{t}_j)\Bigg|$$

$$+\sup_{\mathbf{x}\in D}\frac{\lambda_2}{n}\sum_{j=1}^{n}\Bigg|\Bigg(\lambda_1\int_D k_1(\mathbf{x}-\mathbf{y})\overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j)d\mathbf{y}$$

$$-\frac{\lambda_1}{n}\sum_{i=1}^{n}k_1(\mathbf{x}-\mathbf{t}_i)\overrightarrow{k_2}(\mathbf{t}_i-\mathbf{t}_j)\Bigg).\nabla u(t_j)\Bigg|$$

$$=: \sup_{\mathbf{x}\in D}\frac{\lambda_1}{n}\sum_{j=1}^{n}|i|+\sup_{\mathbf{x}\in D}\frac{\lambda_2}{n}\sum_{j=1}^{n}|ii|$$

and

$$\|III+IV\| = \sup_{\mathbf{x}\in D} \frac{\lambda_1}{n} \sum_{j=1}^{n} \Bigg| \lambda_2 \int_D \overrightarrow{k_2}(\mathbf{x}-\mathbf{y}).\nabla k_1(\mathbf{y}-\mathbf{t}_j)d\mathbf{y} - \frac{\lambda_2}{n}\sum_{i=1}^{n} \overrightarrow{k_2}(\mathbf{x}-t_i).\nabla k_1(\mathbf{t}_i-\mathbf{t}_j)\Bigg|$$

$$+ \sup_{\mathbf{x}\in D} \frac{\lambda_2}{n} \sum_{j=1}^{n} \Bigg| \Bigg(\lambda_2 \int_D \overrightarrow{k_2}(\mathbf{x}-\mathbf{y}).\nabla \overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j)d\mathbf{y} - \frac{\lambda_2}{n}\sum_{i=1}^{n} \overrightarrow{k_2}(\mathbf{x}-\mathbf{t}_i).\nabla \overrightarrow{k_2}(\mathbf{t}_i-\mathbf{t}_j)\Bigg).\nabla u(\mathbf{t}_j)\Bigg|$$

$$=: \sup_{\mathbf{x}\in D} \frac{\lambda_1}{n} \sum_{j=1}^{n} |iii| + \sup_{\mathbf{x}\in D} \frac{\lambda_2}{n} \sum_{j=1}^{n} |iv|$$

where

$$i = \lambda_1 \left(\int_D k_1(\mathbf{x}-\mathbf{y}) k_1(\mathbf{y}-\mathbf{t}_j) d\mathbf{y} - \frac{1}{n}\sum_{i=1}^{n} k_1(\mathbf{x}-\mathbf{t}_i)k_1(\mathbf{t}_i-\mathbf{t}_j)\right)$$

$$= -\lambda_1 \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \int_D k_1(\mathbf{x}-\mathbf{y}) k_1(\mathbf{y}-\mathbf{t}_j) e^{-2\pi i \mathbf{h}.\mathbf{y}} d\mathbf{y}$$

$$= -\lambda_1 \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \int_D \left(\sum_{\mathbf{l}\in\mathbb{Z}^d} \hat{k}_1(\mathbf{l}) e^{2\pi i \mathbf{l}.(\mathbf{x}-\mathbf{y})}\right) k_1(\mathbf{y}-\mathbf{t}_j) e^{-2\pi i \mathbf{h}.\mathbf{y}} d\mathbf{y}$$

$$= -\lambda_1 \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \hat{k}_1(\mathbf{l}) e^{2\pi i \mathbf{l}.\mathbf{x}} \int_D k_1(\mathbf{y}-\mathbf{t}_j) e^{-2\pi i (\mathbf{h}+\mathbf{l}).\mathbf{y}} d\mathbf{y}$$

$$= -\lambda_1 \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \hat{k}_1(\mathbf{l}) e^{2\pi i \mathbf{l}.\mathbf{x}} \hat{k}_1(\mathbf{h}+\mathbf{l}) e^{-2\pi i (\mathbf{h}+\mathbf{l})\mathbf{t}_j},$$

$$ii = \lambda_1 \left(\int_D k_1(\mathbf{x}-\mathbf{y}) \overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j) \nabla u(\mathbf{t}_j) d\mathbf{y} - \frac{1}{n}\sum_{i=1}^{n} k_1(\mathbf{x}-\mathbf{t}_i)\overrightarrow{k_2}(\mathbf{t}_i-\mathbf{t}_j).\nabla u(\mathbf{t}_j)\right)$$

$$= -\lambda_1 \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \int_D k_1(x-y) \overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j).\nabla u(\mathbf{t}_j) e^{-2\pi i \mathbf{h}.\mathbf{y}} d\mathbf{y}$$

$$= -\lambda_1 \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \int_D \Big(\sum_{\mathbf{l}\in\mathbb{Z}^d} \hat{k}_1(\mathbf{l}) e^{2\pi i \mathbf{l}.(\mathbf{x}-\mathbf{y})}\Big) \overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j).\nabla u(\mathbf{t}_j) e^{-2\pi i \mathbf{h}.\mathbf{y}} d\mathbf{y}$$

$$= -\lambda_1 \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \hat{k}_1(\mathbf{l}) e^{2\pi i \mathbf{l}.\mathbf{x}} \int_D \overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j).\nabla u(\mathbf{t}_j) e^{-2\pi i (\mathbf{h}+\mathbf{l})\cdot\mathbf{y}} d\mathbf{y}$$

$$= -\lambda_1 \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \hat{k}_1(\mathbf{l}) e^{2\pi i \mathbf{l}.\mathbf{x}} \hat{k}_3(\mathbf{h}+\mathbf{l}) e^{-2\pi i (\mathbf{h}+\mathbf{l}).\mathbf{t}_j},$$

$$iii = \lambda_2 \Big(\int_D \overrightarrow{k_2}(\mathbf{x}-\mathbf{y}).\nabla k_1(\mathbf{y}-\mathbf{t}_j) d\mathbf{y} - \frac{1}{n}\sum_{i=1}^{n} \overrightarrow{k_2}(\mathbf{x}-\mathbf{t}_i).\nabla k_1(\mathbf{t}_i-\mathbf{t}_j)\Big)$$

$$= -\lambda_2 \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \int_D \overrightarrow{k_2}(\mathbf{x}-\mathbf{y}).\nabla k_1(\mathbf{y}-\mathbf{t}_j) e^{-2\pi i \mathbf{h}.\mathbf{y}} d\mathbf{y}$$

$$= -\lambda_2 \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \int_D \Big(\sum_{\mathbf{l}\in\mathbb{Z}^d} \overrightarrow{\hat{k}_2}(\mathbf{l}) e^{2\pi i \mathbf{l}.(\mathbf{x}-\mathbf{y})}\Big) \nabla k_1(\mathbf{y}-\mathbf{t}_j) e^{-2\pi i \mathbf{h}.\mathbf{y}} d\mathbf{y}$$

$$= -\lambda_2 \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in Z^d} \overrightarrow{\hat{k}_2}(\mathbf{l}) e^{2\pi i \mathbf{l}.\mathbf{x}} \int_D .\nabla k_1(\mathbf{y}-\mathbf{t}_j) e^{-2\pi i (\mathbf{h}+\mathbf{l}).\mathbf{y}} d\mathbf{y}$$

$$= -\lambda_2 \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \overrightarrow{\hat{k}_2}(\mathbf{l}) e^{2\pi i \mathbf{l}.\mathbf{x}} .\nabla \hat{k}_1(\mathbf{h}+\mathbf{l}) e^{-2\pi i (\mathbf{h}+\mathbf{l}).\mathbf{t}_j},$$

$$iv = \lambda_2 \Big(\int_D \overrightarrow{k_2}(\mathbf{x}-\mathbf{y}).\nabla\overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j).\nabla u(\mathbf{t}_j) d\mathbf{y}$$

$$- \frac{1}{n}\sum_{i=1}^{n} \overrightarrow{k_2}(\mathbf{x}-\mathbf{t}_i).\nabla\overrightarrow{k_2}(\mathbf{t}_i-\mathbf{t}_j)\nabla u(\mathbf{t}_j)\Big)$$

$$= -\lambda_2 \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \int_D \overrightarrow{k_2}(\mathbf{x}-\mathbf{y}).\nabla\overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j).\nabla u(\mathbf{t}_j) e^{-2\pi i \mathbf{h}.\mathbf{y}} d\mathbf{y}$$

$$= -\lambda_2 \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \int_D \Big(\sum_{\mathbf{l}\in\mathbb{Z}^d} \overrightarrow{\hat{k}_2}(\mathbf{l}) e^{2\pi i \mathbf{l}.(\mathbf{x}-\mathbf{y})}\Big).\nabla\overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j).\nabla u(\mathbf{t}_j) e^{-2\pi i \mathbf{h}.\mathbf{y}} d\mathbf{y}$$

$$= -\lambda_2 \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \overrightarrow{\hat{k}_2}(\mathbf{l}) e^{2\pi i \mathbf{l}.\mathbf{x}}. \int_D \nabla \overrightarrow{k_2}(\mathbf{y}-\mathbf{t}_j)\nabla u(\mathbf{t}_j) e^{-2\pi i(\mathbf{h}+\mathbf{l}).\mathbf{y}} d\mathbf{y}$$

$$= -\lambda_2 \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \overrightarrow{\hat{k}_2}(\mathbf{l}) e^{2\pi i \mathbf{l}.\mathbf{x}}. \overrightarrow{\hat{k}}_4(\mathbf{h}+\mathbf{l}) e^{-2\pi i(\mathbf{h}+\mathbf{l})t_j}.$$

Therefore we conclude

$$\begin{aligned}
&\|(\widetilde{K}-\widetilde{K}_n)\widetilde{K}_n\| \\
&\le \frac{\lambda_1^2}{n}\sup_{\mathbf{x}\in D}\sum_{j=1}^n \Bigg| \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \hat{k}_1(\mathbf{l})\hat{k}_1(\mathbf{h}+\mathbf{l}) e^{2\pi i \mathbf{l}.\mathbf{x}} e^{-2\pi i(\mathbf{h}+\mathbf{l}).\mathbf{t}_j}\Bigg| \\
&\quad + \frac{\lambda_1\lambda_2}{n}\sup_{\mathbf{x}\in D}\sum_{j=1}^n \Bigg| \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \hat{k}_1(\mathbf{l})\hat{k}_3(\mathbf{h}+\mathbf{l}) e^{2\pi i \mathbf{l}.\mathbf{x}} e^{-2\pi i(\mathbf{h}+\mathbf{l}).\mathbf{t}_j}\Bigg| \\
&\quad + \frac{\lambda_1\lambda_2}{n}\sup_{\mathbf{x}\in D}\sum_{j=1}^n \Bigg| \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \overrightarrow{\hat{k}_2}(\mathbf{l}).\nabla\hat{k}_1(\mathbf{h}+\mathbf{l}) e^{2\pi i \mathbf{l}.\mathbf{x}} e^{-2\pi i(\mathbf{h}+\mathbf{l}).\mathbf{t}_j}\Bigg| \\
&\quad + \frac{\lambda_2^2}{n}\sup_{\mathbf{x}\in D}\sum_{j=1}^n \Bigg| \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \overrightarrow{\hat{k}_2}(\mathbf{l}).\overrightarrow{\hat{k}}_4(\mathbf{h}+\mathbf{l}) e^{2\pi i \mathbf{l}.\mathbf{x}} e^{-2\pi i(\mathbf{h}+\mathbf{l}).\mathbf{t}_j}\Bigg| \\
&\le \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d-\{0\}\\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod n}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \big(\lambda_1^2|\hat{k}_1(\mathbf{l})||\hat{k}_1(\mathbf{h}+\mathbf{l})| + \lambda_1\lambda_2|\hat{k}_1(\mathbf{l})||\hat{k}_3(\mathbf{h}+\mathbf{l})| \\
&\qquad\qquad + \lambda_1\lambda_2|\overrightarrow{\hat{k}_2}(\mathbf{l})\cdot\nabla\hat{k}_1(\mathbf{h}+\mathbf{l})| + \lambda_2^2|\overrightarrow{\hat{k}_2}(\mathbf{l})\cdot\overrightarrow{\hat{k}}_4(\mathbf{h}+\mathbf{l})|\big) \\
&\le \big(\lambda_1^2\|k +_1\|_{\mathbb{H}}^2 + \lambda_1\lambda_2\|k_1\|_{\mathbb{H}}\|\overrightarrow{k_2}\|_{\mathbb{H},\infty} + \lambda_1\lambda_2\|\overrightarrow{k_2}\|_{\mathbb{H},\infty}\|\nabla k_1\|_{\mathbb{H},\infty} \\
&\qquad + \lambda_2^2|\overrightarrow{k_2}\|_{\mathbb{H}}\|\nabla\overrightarrow{k_2}\|_{\mathbb{H}}\big) S_{n,d}(\mathbf{z}),
\end{aligned}$$

thus we have

$$\|(\widetilde{K}-\widetilde{K}_n)\widetilde{K}_n\| \le (\lambda_1^2\beta^2 + 2\lambda_1\lambda_2\beta^2 + \lambda_2^2\beta^2) S_{n,d}(\mathbf{z}),$$

and we write

$$\Delta_n := \|\widetilde{K}^{-1}\|\cdot\|(\widetilde{K}-\widetilde{K}_n)\widetilde{K}_n\| \le \mu(\lambda_1+\lambda_2)^2\beta^2 S_{n,d}(\mathbf{z}).$$

Also, if we put $\Delta_n < 1$, then it is sufficient to demand that $S_{n,d}(\mathbf{z}) < \frac{1}{\mu(\lambda_1+\lambda_2)^2\beta^2}$. Hence we write the following inequality:

$$\begin{aligned}
\|\widetilde{K}_n^{-1}\| &\leq \frac{1 + \|\widetilde{K}^{-1}\|(\lambda_1\|k_1\|_{\mathbb{H}} + \lambda_2\|k_3\|_{\mathbb{H}})\prod_{j=1}^{d}(1+2\zeta(\alpha)\gamma_j)}{1-\Delta_n} \\
&\leq \frac{1+\mu(\lambda_1+\lambda_2)\beta\prod_{j=1}^{d}(1+2\zeta(\alpha)\gamma_j)}{1-\mu\beta^2(\lambda_1+\lambda_2)^2 S_{n,d}(\mathbf{z})} \\
&\leq \frac{1+\mu\beta(\lambda_1+\lambda_2)}{1-\mu\beta^2(\lambda_1+\lambda_2)^2 S_{n,d}(\mathbf{z})}\prod_{j=1}^{d}(1+2\zeta(\alpha)\gamma_j).
\end{aligned}$$

On the other hand, from (C.1) we have

$$\begin{aligned}
\|u-u_n\|_{\sup} &\leq \|\widetilde{K}_n^{-1}\|\cdot\|(\widetilde{K}-\widetilde{K}_n)u\|_{\sup} \\
&\leq \frac{1+\mu\beta(\lambda_1+\lambda_2)}{1-\mu\beta^2(\lambda_1+\lambda_2)^2 S_{n,d}(\mathbf{z})}\prod_{j=1}^{d}(1+2\zeta(\alpha)\gamma_j)\|\widetilde{K}-\widetilde{K}_n)u\|_{\sup} \\
&\leq \frac{1+\mu\beta(\lambda_1+\lambda_2)}{1-\mu\beta^2(\lambda_1+\lambda_2)^2 S_{n,d}(\mathbf{z})} \\
&\qquad \times\prod_{j=1}^{d}(1+2\zeta(\alpha)\gamma_j)\big(\lambda_1\|\widetilde{K}^{-1}\|\cdot\|\overrightarrow{a_1}\cdot\nabla u - g\|_{\mathbb{H}}\|k_1\|_{\mathbb{H}} \\
&\qquad\qquad + \lambda_2\|\nabla u\|_{\mathbb{H},\infty}\|\overrightarrow{k_2}\|_{\mathbb{H},\infty}\big)S_{n,d}(\mathbf{z}) \\
&\leq \frac{(1+\mu\beta(\lambda_1+\lambda_2))\prod_{j=1}^{d}(1+2\zeta(\alpha)\gamma_j)}{1-\mu\beta^2(\lambda_1+\lambda_2)^2 S_{n,d}(\mathbf{z})}(\lambda_1\mu+\lambda_2)\beta S_{n,d}(\mathbf{z}).
\end{aligned}$$

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